Combinatorial problems on closed and privileged words

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Basic Definitions: Bordered and Unbordered words

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The word `alfalfa` is bordered. It has two borders `alfa`, and `a`.

The word `bordered` is in fact unbordered. It has no valid borders.

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The English word `bonobo` is a closed word, closed by `bo`. Since $|\texttt{bo}| > 1$ and `bo` is unbordered and therefore not privileged, we have that `bonobo` is not privileged.

The French word `entente` is closed by `ente`. Furthermore `ente` is closed by `e`. But $|\texttt{e}| \leq 1$, so `ente` is privileged and therefore so is `entente`.

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- A **period** of a word $w = w_1 w_2 \cdots w_n$ is an integer $p \leq n$ such that $w_i = w_{i+p}$ for all $1 \leq i \leq n - p$.
- A length-n word is said to be **periodic** if it has a period of length $\leq n/2$.

In the analysis of DNA sequences, and other long words, the minimal period is generally much larger than half the length of the word.

Periodic-like words were considered as a generalization of periodic words that preserve some features of periodic words.

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Closed words are also known as **complete first return words**.

- A **complete first return** to a word u is a word that starts and ends with u, and contains exactly two occurrences of u.
- A word is closed if it is a complete first return to one of its non-empty proper prefixes.

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Privileged words were originally defined by Kellendonk et al. (2011) in the context of dynamical systems and discrete geometry by iterating the definition of a complete first return.

A word is privileged if and only if it is the empty word, a single letter, or a complete first return to a shorter privileged word.

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For the rest of this talk, we implicitly assume that all words are over a k-letter alphabet for some $k \geq 2$.

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Every privileged word is also closed, so $P_k(n) \leq C_k(n)$.

In 2016, Forsyth et al. showed that

$$P_2(n) \in \Omega\left(\frac{2^n}{n^2}\right).$$

In 2018, Nicholson and Rampersad showed that

$$P_k(n) \in \Omega\left(\frac{k^n}{n(\log_k(n))^2}\right).$$

Problem: Enumeration

In 2020, Rukavicka proved the very first nontrivial upper bounds on privileged and closed words. He showed that

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In 2022, Rukavicka improved his upper bound on privileged words by showing that for every $j \geq 3$,

$$P_k(n) \in O\left(\frac{k^n \sqrt{\ln n}}{\sqrt{n}} \ln^{\circ j}(n) \prod_{i=2}^{j-1} \sqrt{\ln^{\circ i}(n)}\right)$$

where $\ln^{\circ 0}(n) = n$ and $\ln^{\circ j}(n) = \ln(\ln^{\circ j-1}(n))$.

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For c large enough we have that u does not occur in w' with high probability, so w is closed.

There are at least approximately $k^{n-\ell} \in \Theta\left(\frac{k^n}{n}\right)$ length-n closed words.

Enumeration Results

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Let $k \geq 2$ be an integer. Let $\log_k^{\circ 0}(n) = n$ and $\log_k^{\circ j}(n) = \log_k(\log_k^{\circ j-1}(n))$ for $j \geq 1$.

For all $j \geq 0$, we have

$$P_k(n) \in \Omega\left(\frac{k^n}{n \log_k^{\circ j}(n) \prod_{i=1}^{j} \log_k^{\circ i}(n)}\right)$$

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D. Gabrić, Asymptotic bounds for the number of closed and privileged words, E-JC, Vol. 32 (2024) #P2.45

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Therefore, there are $\Omega\left(\frac{k^n}{n}\right)$ length-n closed words.

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Let $C_k(n, t)$ denote the number of length-n closed words that have a longest border of length t.

We have

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We consider two cases, one where longest border is of length $t > \lfloor n/2 \rfloor$ and the other where the length is $t \leq \lfloor n/2 \rfloor$.

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Thus, for any length-n closed word w closed by a length-t word u, we can write $w = uvu$.

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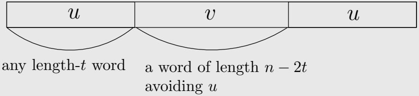
The word u can be any length-t word, and the word v is a word of length $n - 2t$ that does not contain u as a factor.

- The word v has some extra constraints which are unimportant in proving an upper bound.



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Let $A_k(n, u)$ denote the number of length-n words avoiding u as a factor.

$$\underbrace{\boxed{\quad u \quad}}_{k^t \text{ such words}}\underbrace{\boxed{\quad v \quad}}_{A_k(n-2t,\,u) \text{ such words}}\boxed{\quad u \quad}$$

We can upper bound $C_k(n, t)$ by $\sum_{|u|=t} A_k(n - 2t, u)$.

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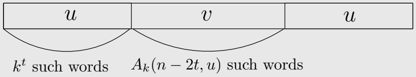

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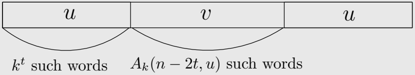

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A crucial result by Guibas and Odlyzko (1981) shows that $A_k(n,u) \le A_k(n,0^{|u|})$ for all $n \ge 1$.

Which means

$$C_k(n,t) \le k^t A_k(n-2t,0^t).$$

Proof Sketch: Closed words

Upper Bound

For $t = 1$, we have $A_k(n, 0^t) = (k-1)^n$ for all $n \geq 0$

For $t \geq 2$, the sequence $(A_k(n, 0^t))_{n \geq 0}$ is exactly the t-bonacci sequence, which has the following two well-known recurrences:

$$A_k(n, 0^t) = \begin{cases} k^n, & n < t; \\ (k-1) \sum_{i=1}^{t} A_k(n-i, 0^t), & n \geq t. \end{cases}$$

$$A_k(n, 0^t) = \begin{cases} k^n, & n < t; \\ k^n - 1, & n = t; \\ kA_k(n-1, 0^t) - A_k(n-1-t, 0^t), & n > t. \end{cases}$$

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Using these recurrences, one can show that $A_k(n, 0^t) \leq (k - (k-1)k^{-t-1})^n$ for all $t \geq 2$, $n \geq 1$.

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We end up with

$$C_k(n) = \sum_{t=1}^{n-1} C_k(n,t) \leq (k-1)^n + nk^{\lceil n/2 \rfloor} + \sum_{t=2}^{\lfloor n/2 \rfloor} k^t (k - (k-1)k^{-t-1})^{n-2t}$$

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Using standard techniques, which are uninteresting for the purpose of this talk, one can show that there is some constant $N > 0$ such that

$$C_k(n) \leq c\frac{k^n}{n}$$

for all $n > N$ where $c > 0$ is some constant which depends on k.

Proof Sketch: Privileged words

Since every privileged word is also a closed word, we have that a privileged word is also closed by its largest border (which must be privileged).

We can write, similar to the closed case, that

$$P_k(n) = \sum_{t=1}^{n-1} P_k(n,t)$$

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Lower Bound

Consider a privileged word w of length n with longest border u of length $t \approx \log_k(n)$.

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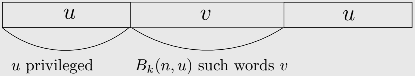
Consider a privileged word w of length n with longest border u of length $t \approx \log_k(n)$.

For n large enough, we can write $w = uvu$ for some word v.



Proof Sketch: Privileged words

Lower Bound

We have

$$P_k(n,t) = \sum_{|u|=t \text{ and } u \text{ privileged}} B_k(n,u).$$

By Nicholson and Rampersad (2018), we have $B_k(n,u) \in \Omega\left(\frac{k^n}{n^2}\right)$ for $|u| \approx \log_k(n)$, and so

$$P_k(n) \geq P_k(n,t) \geq \sum_{|u|=t \text{ and } u \text{ privileged}} c\frac{k^n}{n^2} = cP_k(t)\frac{k^n}{n^2}$$

for all n large enough, where $c > 0$ is some constant, and $t \approx \log_k(n)$.

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for all n large enough, where $c > 0$ is some constant, and $t \approx \log_k(n)$.

We get the lower bound by iteratively applying the above recursive inequality.

We can prove by induction that:

$$P_k(n) \geq c_0 P_k(\log_k(n))\frac{k^n}{n^2} \geq c_1 P_k(\log_k(\log_k(n)))\frac{k^n}{n(\log_k(n))^2} \geq \dots$$
$$\geq c_j \frac{k^n}{n \log_k^{\circ j}(n) \prod_{i=1}^{j} \log_k^{\circ i}(n)}.$$

Proof Sketch: Privileged words

Upper Bound

Similar to the upper bound proof for closed words, we split the sum

$$P_k(n) = \sum_{t=1}^{n-1} P_k(n,t)$$

into two separate sums, one where $t > \lfloor n/2 \rfloor$ and one where $t \leq \lfloor n/2 \rfloor$.

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The same reasoning as in the closed words case shows that there are at most $nk^{\lceil n/2 \rceil}$ length-n privileged words with a longest border of length $t > \lfloor n/2 \rfloor$.

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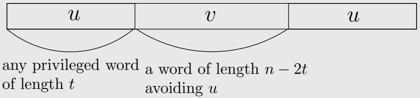
When $t \leq \lfloor n/2 \rfloor$, we have that the longest borders do not overlap.

For any length-n privileged word closed by a length-t privileged word u, we can write $w = uvu$.



Proof Sketch: Privileged words

Upper Bound

$$\underbrace{u}_{P_k(t)\text{ such words}}\ \underbrace{v}_{A_k(n-2t,\,u)\text{ such words}}\ u$$

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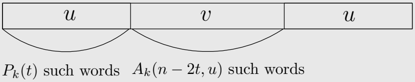
Upper Bound



We already have an upper bound on $A_k(n-2t,u)$ from before, so we get the following recursive upper bound:

$$P_k(n,t) \le \sum_{|u|=t \text{ and } u \text{ privileged}} A_k(n-2t,u) \le \sum_{|u|=t \text{ and } u \text{ privileged}} A_k(n-2t,0^t)$$
$$\le P_k(t)(k-(k-1)k^{-t-1})^{n-2t}.$$

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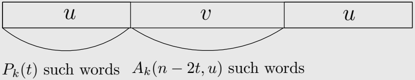

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$$\leq P_k(t)(k-(k-1)k^{-t-1})^{n-2t}.$$

We can show by induction on j, that

$$P_k(n) \leq c_j \frac{k^n}{n\prod_{i=1}^{j}\log_k^{\circ i}(n)}$$

for all n large enough where $c_j > 0$ is some constant, and the base case when $j=0$ is $P_k(n) \leq C_k(n) \in O\left(\frac{k^n}{n}\right)$.

Open Problems

Let ℓ_n denote the smallest positive integer such that $\log_k^{\circ \ell_n + 1}(n) \leq 1$.

Define

$$\log_k^*(n) := \prod_{j=1}^{\ell_n} \log_k^{\circ j}(n).$$

Conjecture

For all $k \geq 2$, we have

$$P_k(n) \in \Theta\left(\frac{k^n}{n \log_k^*(n)}\right).$$

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For all $k \geq 2$, we have

$$P_k(n) \in \Theta\left(\frac{k^n}{n \log_k^*(n)}\right).$$

This should be provable by a careful analysis of the multiplicative constants introduced in the proofs of the upper and lower bounds of $P_k(n)$.

Open Problems

Open Question

Does the limit

$$\lim_{n\to\infty} \frac{C_k(n)n}{k^n}$$

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A **prefix-synchronized code** of length n is a set of codewords of length n that all begin with a fixed prefix u of length t, and every codeword is a prefix of a closed word of length $n+t$ that is closed by u.

A prefix-synchronized code is said to be **maximal** if it contains every possible codeword that begins with the fixed prefix.

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Guibas and Odlyzko (1978) showed that for $k = 2, 3, 4$ a prefix-synchronized code is maximal if the fixed length-t prefix u is unbordered and $t \approx \log_k(n)$.

Open Problems

They showed that the ratio between the size of a maximal prefix-synchronized code of length n and $\frac{k^n}{n}$ oscillates between $0.3679\ldots$, and $0.3465\ldots$ (for $k=2$), $0.3171\ldots$ (for $k=3$), and 0.2911 (for $k=4$).

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However, just looking experimentally, it seems that the ratio $\frac{nC_k(n)}{k^n}$ tends to a constant.

n	$\frac{nC_2(n)}{2^n}$	n	$\frac{nC_3(n)}{3^n}$	n	$\frac{nC_4(n)}{4^n}$
42	1.633719	19	1.044144	14	0.808427
43	1.630123	20	1.039228	15	0.802991
44	1.626657	21	1.034529	16	0.799168
45	1.623312	22	1.030038	17	0.796606
46	1.620082	23	1.025759	18	0.794974
47	1.616962	24	1.021695	19	0.793994
48	1.613948	25	1.017851	20	0.793439
49	1.611034	26	1.014232	21	0.793127
50	1.608218	27	1.010838	22	0.792917
51	1.605495	28	1.007667	23	0.792707

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Definition The **autocorrelation** u of a length-n binary word w is a binary word of length n that has a 1 at position i if and only if w has a border of length $n - i + 1$.

In this definition, borders are allowed to be non-proper.

For example, the word `redelivered` has autocorrelation 10000000100.

Open Problems

There is a recurrence for $B_k(n, u)$ that depends on n and the autocorrelation of u.

This implies that $B_k(n, u) = B_k(n, v)$ if u and v share the same autocorrelation.

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To count closed words in $o(C_k(n))$ time, all we would need to do is

1. Count the number of periodic words of length n, handling the case where the longest border is of length $> \lfloor n/2 \rfloor$.
2. For the case where the longest border is of length $\leq \lfloor n/2 \rfloor$. Generate all autocorrelations of length $\leq \lfloor n/2 \rfloor$, compute $B_k(n, u)$ for every individual autocorrelation, and compute the number of words having every one of these autocorrelations.

Open Problems

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I have no idea how to count privileged words in $o(P_k(n))$ time.

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Assume all closed/privileged words are listed in some order, perhaps lexicographic.

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Ranking: Given a length-n closed (resp. privileged) word, compute its position in the listing.

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Example

Consider the length-5 closed words in lexicographic order.

Closed word	Position
00000	1
00100	2
01001	3
01010	4
01101	5
01110	6

Closed word	Position
10001	7
10010	8
10101	9
10110	10
11011	11
11111	12

Ranking: The position of the closed word 01101 is 5.
Unranking: The closed word at position 8 is 10010.

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Then ranking a length-n closed word w involves summing this recurrence over at most kn words of the form va where v is a prefix of w and $a < w[|v|+1]$.

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Once ranking is solved, unranking can also be solved by repeated ranking.

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The first algorithms to rank/unrank bordered and unbordered words were recently presented:

D. Gabrić. Ranking and unranking bordered and unbordered words. Information Processing Letters. Vol. 184 (2024) 106452

J. Radoszewski, W. Rytter, T. Waleń. Faster Algorithms for Ranking/Unranking Bordered and Unbordered Words. String Processing and Information Retrieval. SPIRE 2024. Lecture Notes in Computer Science, vol 14899. (2025)

Generation

Generation: Compute every length-n closed (resp. privileged) word in lexicographic order (or any order) in time proportional to $C_k(n)$ (resp. $P_k(n)$), i.e., in constant-amortized time per word.

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The failure function from the KMP string matching algorithm computes the length of the longest border of every prefix of a word, and can be computed in $O(n)$ time.

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Can we do better?

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All length-n closed words can be generated in constant-amortized time per word in lexicographic order.

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The failure function can be computed symbol by symbol from left-to-right.

Start recursively generating all words in lexicographic order and compute the failure function in parallel.

Based on the failure function, infer whether a prefix can be uniquely extended to a closed word.

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Using the algorithm for closed words, one can compute all privileged words in amortized time between $\Omega(\log_k(n))$ and $O(\log_k^2(n))$ per word.

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All we need to do is compute this algorithm in parallel to the recursive algorithm to generate all closed words.

I have not figured out how to improve this algorithm to run in constant-amortized time per word.

Question

Is it possible to generate all length-n privileged words in constant-amortized time per word? In lexicographic order? In any order?

Thank you!