Binomial Coefficients of Multidimensional Arrays

Mehdi Golafshan, Michel Rigo

June 30, 2025

LIÈGE université

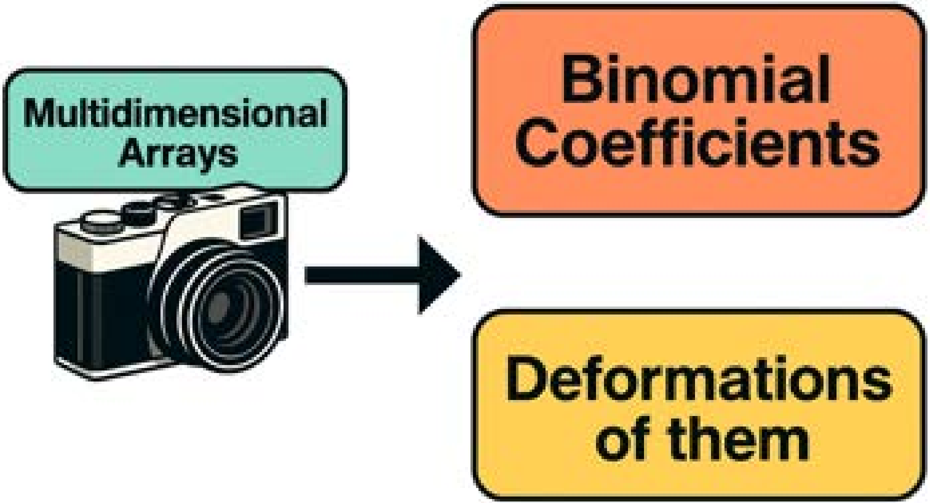
WORDS 2025, Nancy (?)

WORDS 2025, Nancy (?)

QR Code — a 2-Dimensional Barcode

- Black/white modules form a finite $n \times n$ binary picture.
- Encodes data + Reed–Solomon error-correction in the 2-D layout.
- Scanners act like automata recognising the picture language of valid codes.

Snapshot


Multidimensional Arrays
Binomial Coefficients
Deformations of them

Notation

Notation

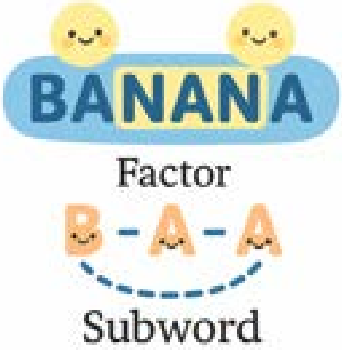
- $|w| = \#$ of letters in w
- $|w|_{\mathtt{a}} = \#$ of occurrences of the letter a in w
- In a word

  factor: subsequence of consecutive letters
  subword: subsequence of letters

Notation

- $|w| = \#$ of letters in w
- $|w|_{\mathtt{a}} = \#$ of occurrences of the letter a in w
- In a word

  factor: subsequence of consecutive letters
  subword: subsequence of letters



Multidimensional (Finite) Words

- $\mathcal{A}$: finite alphabet
- $k \geqslant 1$

- $\mathcal{A}$: finite alphabet
- $k \geqslant 1$

Definition (k-array)

A **multidimensional word** (k-**array**) is a finite hyper-rectangular array of symbols from $\mathcal{A}$.

Multidimensional (Finite) Words

- $\mathcal{A}$: finite alphabet
- $k \geqslant 1$

Definition (k-array)

A **multidimensional word** (*k*-**array**) is a finite hyper-rectangular array of symbols from $\mathcal{A}$.

Binomial Coefficients (Integers to Words)

Binomial Coefficients (Integers to Words)

$$\binom{\mathtt{a}^m}{\mathtt{a}^n} = \binom{m}{n}, \quad m, n \in \mathbb{N}$$

$$\binom{\mathtt{a}^m}{\mathtt{a}^n} = \binom{m}{n}, \quad m, n \in \mathbb{N}$$

- Integers to Words: $\mathcal{A} = \{\mathtt{a}\}$
- What about for an arbitrary alphabet $\mathcal{A}$?

Binomial Coefficients (Integers to Words)

$$\binom{\mathtt{a}^m}{\mathtt{a}^n} = \binom{m}{n}, \quad m, n \in \mathbb{N}$$

- Integers to Words: $\mathcal{A} = \{\mathtt{a}\}$
- What about for an arbitrary alphabet $\mathcal{A}$?

- $u, v \in \mathcal{A}^*$

Definition (binomial coefficient of word)

The **binomial coefficient** $\binom{u}{v}$ is the number of occurrences of v as a subword of u.

- $u, v \in \mathcal{A}^*$

Definition (binomial coefficient of word)

The **binomial coefficient** $\binom{u}{v}$ is the number of occurrences of v as a subword of u.

- $u, v \in \mathcal{A}^*$

Definition (binomial coefficient of word)

The **binomial coefficient** $\binom{u}{v}$ is the number of occurrences of v as a subword of u.

Example (1)

- $u = 101001$
- $v = 101$

101001 101001 101001
101001 101001 101001

$$\binom{u}{v} = \binom{101001}{101} = 6$$

Zero–Dimention

- $\mathtt{u}, \mathtt{v} \in \mathcal{A} \cup \{\varepsilon\}$ (single symbol or empty): zero-dimensional words

$$\delta_{\mathtt{u},\mathtt{v}} = \binom{\mathtt{u}}{\mathtt{v}} = \begin{cases} 1, & \mathtt{u} = \mathtt{v}, \\ 0, & \text{otherwise.} \end{cases}$$

  - $\delta_{\mathtt{u},\mathtt{v}}$: Kronecker symbol

Zero–Dimention

- $\mathtt{u}, \mathtt{v} \in \mathcal{A} \cup \{\varepsilon\}$ (single symbol or empty): zero-dimensional words

$$\delta_{\mathtt{u},\mathtt{v}} = \binom{\mathtt{u}}{\mathtt{v}} = \begin{cases} 1, & \mathtt{u} = \mathtt{v}, \\ 0, & \text{otherwise.} \end{cases}$$

  - $\delta_{\mathtt{u},\mathtt{v}}$: Kronecker symbol

One–Dimention

- $u = \mathtt{u}_1\mathtt{u}_2 \cdots \mathtt{u}_k \in \mathcal{A}^*$
- $v = \mathtt{v}_1\mathtt{v}_2 \cdots \mathtt{v}_\ell \in \mathcal{A}^*$
- $u_i,\ v_j \in \mathcal{A}$

$$\binom{u}{v} = \sum_{1 \leqslant i_1 < \cdots < i_\ell \leqslant k} \delta_{\mathtt{u}_{i_1},\mathtt{v}_1} \cdots \delta_{\mathtt{u}_{i_\ell},\mathtt{v}_\ell}.$$

One–Dimention

- $u = \mathtt{u}_1\mathtt{u}_2\cdots\mathtt{u}_k \in \mathcal{A}^*$
- $v = \mathtt{v}_1\mathtt{v}_2\cdots\mathtt{v}_\ell \in \mathcal{A}^*$
- $u_i,\ v_j \in \mathcal{A}$

$$\binom{u}{v} = \sum_{1\leqslant i_1<\cdots<i_\ell\leqslant k} \delta_{\mathtt{u}_{i_1},\mathtt{v}_1}\cdots\delta_{\mathtt{u}_{i_\ell},\mathtt{v}_\ell}.$$

- $M = \begin{pmatrix} M_1 & \cdots & M_m \end{pmatrix} \in \mathcal{A}^{**}$
- $P = \begin{pmatrix} P_1 & \cdots & P_p \end{pmatrix} \in \mathcal{A}^{**}$

One–Dimention

- $u = \mathtt{u}_1\mathtt{u}_2\cdots\mathtt{u}_k \in \mathcal{A}^*$
- $v = \mathtt{v}_1\mathtt{v}_2\cdots\mathtt{v}_\ell \in \mathcal{A}^*$
- $u_i,\ v_j \in \mathcal{A}$

$$\binom{u}{v} = \sum_{1\leqslant i_1<\cdots<i_\ell\leqslant k} \delta_{\mathtt{u}_{i_1},\mathtt{v}_1}\cdots\delta_{\mathtt{u}_{i_\ell},\mathtt{v}_\ell}.$$

- $M = \begin{pmatrix} M_1 & \cdots & M_m \end{pmatrix} \in \mathcal{A}^{**}$
- $P = \begin{pmatrix} P_1 & \cdots & P_p \end{pmatrix} \in \mathcal{A}^{**}$

Definition (1)

The **column-binomial coefficient** of M and P is defined by

$$\binom{M}{P} = \sum_{1\leqslant i_1<\cdots<i_p\leqslant m} \binom{M_{i_1}}{P_1}\cdots\binom{M_{i_p}}{P_p}.$$

- $\binom{M_{i_j}}{P_j}$: the classical binomial coefficient of 1-D words

Definition (1)

The **column-binomial coefficient** of M and P is defined by

$$\binom{M}{P} = \sum_{1 \leqslant i_1 < \cdots < i_p \leqslant m} \binom{M_{i_1}}{P_1} \cdots \binom{M_{i_p}}{P_p}.$$

Definition (1)

The **column-binomial coefficient** of M and P is defined by

$$\binom{M}{P} = \sum_{1\leqslant i_1<\cdots<i_p\leqslant m} \binom{M_{i_1}}{P_1} \cdots \binom{M_{i_p}}{P_p}.$$

Example (2)

$$M = \begin{pmatrix} \mathtt{a} & \mathtt{b} & \mathtt{a} & \mathtt{a} \\ \mathtt{b} & \mathtt{a} & \mathtt{a} & \mathtt{a} \\ \mathtt{a} & \mathtt{a} & \mathtt{b} & \mathtt{a} \\ \mathtt{a} & \mathtt{b} & \mathtt{b} & \mathtt{a} \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} \mathtt{a} & \mathtt{a} \\ \mathtt{a} & \mathtt{a} \end{pmatrix}$$

Definition (1)

The **column-binomial coefficient** of M and P is defined by

$$\binom{M}{P} = \sum_{1 \leqslant i_1 < \cdots < i_p \leqslant m} \binom{M_{i_1}}{P_1} \cdots \binom{M_{i_p}}{P_p}.$$

Example (2)

$$M = \begin{pmatrix} \mathtt{a} & \mathtt{b} & \mathtt{a} & \mathtt{a} \\ \mathtt{b} & \mathtt{a} & \mathtt{a} & \mathtt{a} \\ \mathtt{a} & \mathtt{a} & \mathtt{b} & \mathtt{a} \\ \mathtt{a} & \mathtt{b} & \mathtt{b} & \mathtt{a} \end{pmatrix} \qquad \text{and} \qquad P = \begin{pmatrix} \mathtt{a} & \mathtt{a} \\ \mathtt{a} & \mathtt{a} \end{pmatrix}$$

$$\binom{M_1}{P_1} = 3, \quad \binom{M_2}{P_1} = 1, \quad \binom{M_3}{P_1} = 1, \quad \binom{M_4}{P_1} = 6,$$

$$\binom{M_1}{P_2} = 3, \quad \binom{M_2}{P_2} = 1, \quad \binom{M_3}{P_2} = 1, \quad \binom{M_4}{P_2} = 6.$$

Definition (1)

The **column-binomial coefficient** of M and P is defined by

$$\binom{M}{P} = \sum_{1 \leqslant i_1 < \cdots < i_p \leqslant m} \binom{M_{i_1}}{P_1} \cdots \binom{M_{i_p}}{P_p}.$$

Example (2)

$$M = \begin{pmatrix} \mathtt{a} & \mathtt{b} & \mathtt{a} & \mathtt{a} \\ \mathtt{b} & \mathtt{a} & \mathtt{a} & \mathtt{a} \\ \mathtt{a} & \mathtt{a} & \mathtt{b} & \mathtt{a} \\ \mathtt{a} & \mathtt{b} & \mathtt{b} & \mathtt{a} \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} \mathtt{a} & \mathtt{a} \\ \mathtt{a} & \mathtt{a} \end{pmatrix}$$

$$\binom{M_1}{P_1} = 3, \quad \binom{M_2}{P_1} = 1, \quad \binom{M_3}{P_1} = 1, \quad \binom{M_4}{P_1} = 6,$$

$$\binom{M_1}{P_2} = 3, \quad \binom{M_2}{P_2} = 1, \quad \binom{M_3}{P_2} = 1, \quad \binom{M_4}{P_2} = 6.$$

$$\binom{M}{P} = 3(1+1+6) + 1(1+6) + 1(6) = \boxed{37}$$

Binomial Coefficients (k-Dimension)

- $M = \begin{pmatrix} M_1 & \cdots & M_m \end{pmatrix} \in \mathcal{A}^{**}$
- $P = \begin{pmatrix} P_1 & \cdots & P_p \end{pmatrix} \in \mathcal{A}^{**}$

Definition (1)

The **column-binomial coefficient** of M and P is defined by

$$\binom{M}{P} = \sum_{1 \leqslant i_1 < \cdots < i_p \leqslant m} \binom{M_{i_1}}{P_1} \cdots \binom{M_{i_p}}{P_p}.$$

Binomial Coefficients (k-Dimension)

- $M = \begin{pmatrix} M_1 & \cdots & M_m \end{pmatrix} \in \mathcal{A}^{**}$
- $P = \begin{pmatrix} P_1 & \cdots & P_p \end{pmatrix} \in \mathcal{A}^{**}$

Definition (1)

The **column-binomial coefficient** of M and P is defined by

$$\binom{M}{P} = \sum_{1 \leqslant i_1 < \cdots < i_p \leqslant m} \binom{M_{i_1}}{P_1} \cdots \binom{M_{i_p}}{P_p}.$$

- $M \in \mathcal{A}^{d_1 \times \cdots \times d_k}$
- $P \in \mathcal{A}^{e_1 \times \cdots \times e_k}$
- $(e_k \leqslant d_k)$.

Binomial Coefficients (k-Dimension)

- $M = \begin{pmatrix} M_1 & \cdots & M_m \end{pmatrix} \in \mathcal{A}^{**}$
- $P = \begin{pmatrix} P_1 & \cdots & P_p \end{pmatrix} \in \mathcal{A}^{**}$

Definition (1)

The **column-binomial coefficient** of M and P is defined by

$$\binom{M}{P} = \sum_{1 \leqslant i_1 < \cdots < i_p \leqslant m} \binom{M_{i_1}}{P_1} \cdots \binom{M_{i_p}}{P_p}.$$

- $M \in \mathcal{A}^{d_1 \times \cdots \times d_k}$
- $P \in \mathcal{A}^{e_1 \times \cdots \times e_k}$
- $(e_k \leqslant d_k)$.

Definition (2)

The **slice binomial coefficient** of M and P is given by

$$\binom{M}{P} = \sum_{1 \leqslant i_1 < \cdots < i_{e_k} \leqslant d_k} \binom{M_{i_1}}{P_{e_1}} \cdots \binom{M_{i_{e_k}}}{P_{e_k}}$$

- $M \in \mathcal{A}^{d_1 \times \cdots \times d_k}$
- $P \in \mathcal{A}^{e_1 \times \cdots \times e_k}$
- $(e_k \leqslant d_k)$.

Definition (2)

The **slice binomial coefficient** of M and P is given by

$$\binom{M}{P} = \sum_{1 \leqslant i_1 < \cdots < i_{e_k} \leqslant d_k} \binom{M_{i_1}}{P_{e_1}} \cdots \binom{M_{i_{e_k}}}{P_{e_k}}$$

- M_i: the $(k-1)$-dimensional slice obtained by fixing the last coordinate of M to i
- $\binom{M_{i_j}}{P_{e_j}}$: a binomial coefficient of $(k-1)$-dimensional slice arrays.

Pascal's Identity (2D-Words)

Proposition (Pascal Identity)

- $u, v \in \mathcal{A}^*$
- $\mathtt{a}, \mathtt{b} \in \mathcal{A}$

$$\binom{u\mathtt{a}}{v\mathtt{b}} = \binom{u}{v\mathtt{b}} + \delta_{\mathtt{a},\mathtt{b}} \binom{u}{v}.$$

Proposition (Pascal Identity)

- $u, v \in \mathcal{A}^*$
- $\mathtt{a}, \mathtt{b} \in \mathcal{A}$

$$\binom{u\mathtt{a}}{v\mathtt{b}} = \binom{u}{v\mathtt{b}} + \delta_{\mathtt{a},\mathtt{b}} \binom{u}{v}.$$

Proposition (G. & Rigo 2025)

- $M \in \mathcal{A}^{r \times m}$
- $P \in \mathcal{A}^{s \times p}$
- $C \in \mathcal{A}^{r \times 1}$
- $D \in \mathcal{A}^{s \times 1}$

$$\binom{M \varobar C}{P \varobar D} = \binom{M}{P \varobar D} + \binom{C}{D}\binom{M}{P}.$$

* $A \varobar B$: concatenation of two arrays (have the same number of rows)

Chu–Vandermonde's Identity (2D-Words)

Proposition (Chu–Vandermonde's Identity)

- $u, v, w \in \mathcal{A}^*$

$$\binom{uv}{w} = \sum_{\substack{x\,y=w \\ x,y\in\mathcal{A}^*}} \binom{u}{x}\binom{v}{y}.$$

Proposition (Chu–Vandermonde's Identity)

- $u, v, w \in \mathcal{A}^*$

$$\binom{uv}{w} = \sum_{\substack{x\,y=w\\ x,y\in\mathcal{A}^*}} \binom{u}{x}\binom{v}{y}.$$

Proposition (G. & Rigo 2025)

- $M \in \mathcal{A}^{r\times m}$
- $N \in \mathcal{A}^{r\times n}$

$$\binom{M \mathbin{\varobar} N}{P} = \sum_{P_1 \mathbin{\varobar} P_2 = P} \binom{M}{P_1}\binom{N}{P_2}.$$

- $M = \begin{pmatrix} M_1 & \cdots & M_m \end{pmatrix} \in \mathcal{A}^{**}$
- $P = \begin{pmatrix} P_1 & \cdots & P_p \end{pmatrix} \in \mathcal{A}^{**}$

- $M = \begin{pmatrix} M_1 & \cdots & M_m \end{pmatrix} \in \mathcal{A}^{**}$
- $P = \begin{pmatrix} P_1 & \cdots & P_p \end{pmatrix} \in \mathcal{A}^{**}$

Definition (3)

Let $\Psi : \mathcal{A}^{m\times 1} \to \mathbb{N}^{(p+1)\times(p+1)}$ be a morphism of monoids such that, for all $k \in \{1,\dots,m\}$ and $i \in \{1,\dots,p+1\}$,

$$[\Psi(M_k)]_{i,j} = \begin{cases} 1, & j = i, \\ \begin{pmatrix} M_k \\ P_i \end{pmatrix}, & j = i+1 \quad (1 \leqslant i \leqslant p), \\ 0, & \text{otherwise.} \end{cases}$$

This is the **column-Parikh matrix** of M_k.

Example (3)

Let

$$M = \begin{pmatrix} \mathtt{a} & \mathtt{b} & \mathtt{a} & \mathtt{a} \\ \mathtt{b} & \mathtt{a} & \mathtt{a} & \mathtt{a} \\ \mathtt{a} & \mathtt{a} & \mathtt{b} & \mathtt{a} \\ \mathtt{a} & \mathtt{b} & \mathtt{b} & \mathtt{a} \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} \mathtt{a} & \mathtt{a} \\ \mathtt{b} & \mathtt{b} \end{pmatrix}.$$

We obtain

$$\Psi\begin{pmatrix} \mathtt{a} \\ \mathtt{b} \\ \mathtt{a} \\ \mathtt{a} \end{pmatrix} \Psi\begin{pmatrix} \mathtt{b} \\ \mathtt{a} \\ \mathtt{a} \\ \mathtt{b} \end{pmatrix} \Psi\begin{pmatrix} \mathtt{a} \\ \mathtt{a} \\ \mathtt{b} \\ \mathtt{b} \end{pmatrix} \Psi\begin{pmatrix} \mathtt{a} \\ \mathtt{a} \\ \mathtt{a} \\ \mathtt{a} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 7 & 14 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{pmatrix}.$$

Example (3)

Let

$$M = \begin{pmatrix} \mathtt{a} & \mathtt{b} & \mathtt{a} & \mathtt{a} \\ \mathtt{b} & \mathtt{a} & \mathtt{a} & \mathtt{a} \\ \mathtt{a} & \mathtt{a} & \mathtt{b} & \mathtt{a} \\ \mathtt{a} & \mathtt{b} & \mathtt{b} & \mathtt{a} \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} \mathtt{a} & \mathtt{a} \\ \mathtt{b} & \mathtt{b} \end{pmatrix}.$$

We obtain

$$\Psi\begin{pmatrix} \mathtt{a} \\ \mathtt{b} \\ \mathtt{a} \\ \mathtt{a} \end{pmatrix} \Psi\begin{pmatrix} \mathtt{b} \\ \mathtt{a} \\ \mathtt{a} \\ \mathtt{b} \end{pmatrix} \Psi\begin{pmatrix} \mathtt{a} \\ \mathtt{a} \\ \mathtt{b} \\ \mathtt{b} \end{pmatrix} \Psi\begin{pmatrix} \mathtt{a} \\ \mathtt{a} \\ \mathtt{a} \\ \mathtt{a} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 7 & 14 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{pmatrix}.$$

Indeed, the entries of the matrix correspond to

$$\binom{M}{(\mathtt{a} \quad \mathtt{b})^{\intercal}} = 7, \quad \binom{M}{(\mathtt{a} \quad \mathtt{b})^{\intercal}} = 7 \quad \text{and} \quad \binom{M}{P} = 14.$$

$$\boxed{\Psi(M) := \Psi(M_1) \cdots \Psi(M_m)}$$

$$\boxed{\Psi(M) := \Psi(M_1) \cdots \Psi(M_m)}$$

Theorem (G. & Rigo 2025)

- $\Psi(M)$: an upper-unitriangular $(p+1) \times (p+1)$ matrix

$\binom{M}{P}$: encoded directly in the single entry $(1, p+1)$ of the unitriangular matrix $\Psi(M)$.

$$\boxed{\Psi(M) := \Psi(M_1)\cdots\Psi(M_m)}$$

Theorem (G. & Rigo 2025)

- $\Psi(M)$: an upper-unitriangular $(p+1)\times(p+1)$ matrix
- For every super-diagonal position (i,j) with $1 \leqslant i < j \leqslant p+1$

$$[\Psi(M)]_{i,j} \;=\; \binom{M}{(P_i\ \ldots\ P_{j-1})}.$$

$\binom{M}{P}$: encoded directly in the single entry $(1, p+1)$ of the unitriangular matrix $\Psi(M)$.

$$\boxed{\Psi(M) := \Psi(M_1)\cdots\Psi(M_m)}$$

Theorem (G. & Rigo 2025)

- $\Psi(M)$: an upper-unitriangular $(p+1)\times(p+1)$ matrix
- For every super-diagonal position (i,j) with $1 \leqslant i < j \leqslant p+1$

$$[\Psi(M)]_{i,j} = \binom{M}{(P_i \ \ldots \ P_{j-1})}.$$

- The upper-right corner satisfies

$$[\Psi(M)]_{1,p+1} = \binom{M}{P}.$$

$$\boxed{\Psi(M) := \Psi(M_1)\cdots\Psi(M_m)}$$

Theorem (G. & Rigo 2025)

- $\Psi(M)$: an upper-unitriangular $(p+1)\times(p+1)$ matrix
- For every super-diagonal position (i,j) with $1 \leqslant i < j \leqslant p+1$

$$[\Psi(M)]_{i,j} = \binom{M}{(P_i \ \ldots \ P_{j-1})}.$$

- The upper-right corner satisfies

$$[\Psi(M)]_{1,p+1} = \binom{M}{P}.$$

$\binom{M}{P}$: encoded directly in the single entry $(1, p+1)$ of the unitriangular matrix $\Psi(M)$.

Hyperrectangle in another Hyperrectanle

- $\mathtt{a} \in \mathcal{A}$

$$\binom{\mathtt{a}^m}{\mathtt{a}^n} = \binom{m}{n}, \quad m, n \in \mathbb{N}$$

Hyperrectangle in another Hyperrectanle

- $\mathtt{a} \in \mathcal{A}$

$$\binom{\mathtt{a}^m}{\mathtt{a}^n} = \binom{m}{n}, \quad m, n \in \mathbb{N}$$

Proposition (G. & Rigo 2025)

- $M \in \mathcal{A}^{r \times m}$

$$\sum_{P \in \mathcal{A}^{s \times p}} \binom{M}{P} = \binom{\mathtt{a}^{r \times m}}{\mathtt{a}^{s \times p}} = \binom{m}{p} \cdot \binom{r}{s}^{p}.$$

- $\mathtt{a} \in \mathcal{A}$

$$\binom{\mathtt{a}^m}{\mathtt{a}^n} = \binom{m}{n}, \quad m, n \in \mathbb{N}$$

Proposition (G. & Rigo 2025)

- $M \in \mathcal{A}^{r \times m}$

$$\sum_{P \in \mathcal{A}^{s \times p}} \binom{M}{P} = \binom{\mathtt{a}^{r \times m}}{\mathtt{a}^{s \times p}} = \binom{m}{p} \cdot \binom{r}{s}^{p}.$$

- $k \geqslant 2$ and $M \in \mathcal{A}^{d_1 \times \cdots \times d_k}$.
- By induction on k

$$\sum_{P \in \mathcal{A}^{e_1 \times \cdots \times e_k}} \binom{M}{P} = \binom{d_k}{e_k} \binom{d_{k-1}}{e_{k-1}}^{e_k} \binom{d_{k-2}}{e_{k-2}}^{e_k e_{k-1}} \cdots \binom{d_1}{e_1}^{e_k e_{k-1} \cdots e_2}.$$

Example (4)

$$\begin{pmatrix} \mathtt{a} & \mathtt{b} \\ \mathtt{a} & \mathtt{a} \end{pmatrix} ⧢ \begin{pmatrix} \mathtt{a} & \mathtt{b} \\ \mathtt{a} & \mathtt{b} \end{pmatrix} = \begin{pmatrix} \mathtt{a} & \mathtt{b} & \mathtt{a} & \mathtt{b} \\ \mathtt{a} & \mathtt{a} & \mathtt{a} & \mathtt{b} \end{pmatrix} + 2 \begin{pmatrix} \mathtt{a} & \mathtt{a} & \mathtt{b} & \mathtt{b} \\ \mathtt{a} & \mathtt{a} & \mathtt{a} & \mathtt{b} \end{pmatrix} + 2 \begin{pmatrix} \mathtt{a} & \mathtt{a} & \mathtt{b} & \mathtt{b} \\ \mathtt{a} & \mathtt{a} & \mathtt{b} & \mathtt{a} \end{pmatrix} + \begin{pmatrix} \mathtt{a} & \mathtt{b} & \mathtt{a} & \mathtt{b} \\ \mathtt{a} & \mathtt{b} & \mathtt{a} & \mathtt{a} \end{pmatrix}.$$

Proposition (Subword-Shuffle Identity)

- $u \in \mathcal{A}^*$
- $p \in \mathcal{A}^*$
- $q \in \mathcal{A}^*$

$$\binom{u}{p}\binom{u}{q} = \sum_{r \in \mathcal{A}^*} \left\langle p \sqcup\!\sqcup q,\ r \right\rangle \binom{u}{r}.$$

Proposition (Subword-Shuffle Identity)

- $u \in \mathcal{A}^*$
- $p \in \mathcal{A}^*$
- $q \in \mathcal{A}^*$

$$\binom{u}{p}\binom{u}{q} = \sum_{r \in \mathcal{A}^*} \langle p \sqcup\!\sqcup q,\ r \rangle \binom{u}{r}.$$

Proposition (G. & Rigo 2025)

- $M \in \mathcal{A}^{r \times m}$
- $P \in \mathcal{A}^{s \times p}$
- $Q \in \mathcal{A}^{s \times q}$

$$\binom{M}{P}\binom{M}{Q} = \sum_{R \in (\mathcal{A}^{s \times 1})^*} \langle P \sqcup\!\sqcup Q, R \rangle \binom{M}{R}$$

$$+ \sum_{\substack{1 \leqslant i_1 < \cdots < i_p \leqslant m \\ 1 \leqslant j_1 < \cdots < j_q \leqslant m \\ \{i_1,\ldots,i_p\} \cap \{j_1,\ldots,j_q\} \neq \varnothing}} \binom{(M_{i_1} \cdots M_{i_p})}{P} \cdot \binom{(M_{j_1} \cdots M_{j_q})}{Q}.$$

Theorem (Manvel's Identity)

- $u, x \in \mathcal{A}^*$
- $k \in \mathbb{N}$
- $|x| \leqslant k \leqslant |u|$

$$\sum_{t \in \mathcal{A}^k} \binom{u}{t} \binom{t}{x} = \binom{|u| - |x|}{k - |x|} \binom{u}{x}.$$

Manvel's Identity (2D-Words)

Theorem (Manvel's Identity)

- $u, x \in \mathcal{A}^*$
- $k \in \mathbb{N}$
- $|x| \leqslant k \leqslant |u|$

$$\sum_{t \in \mathcal{A}^k} \binom{u}{t}\binom{t}{x} = \binom{|u|-|x|}{k-|x|}\binom{u}{x}.$$

Theorem (G. & Rigo 2025)

- $M \in \mathcal{A}^{r \times m}$
- $P \in \mathcal{A}^{s \times p}$

$$\sum_{T \in \mathcal{A}^{q \times t}} \binom{M}{T}\binom{T}{P} = \binom{m-p}{t-p}\binom{r}{q}^{t-p}\binom{r-s}{q-s}^{p}\binom{M}{P}.$$

Let's Recap (I)

Let’s Recap (I)


0
Review of Binomial Coefficients of Words
1
Column and Slice Binomial Coefficients
2
Column Parikh Matrix
3
Column Combinatorial Identities (Pascal, Chu–Vandermonde, Manvel)

q-Deformations

- We choose the recursive definition of q-binomials processed from the left.

- We choose the recursive definition of q-binomials processed from the left.

Definition (Renard, Rigo & Whiteland 2025)

- $u, v \in \mathcal{A}^*$
- $\mathtt{c}, \mathtt{d} \in \mathcal{A}$

$$\binom{\mathtt{c}u}{\mathtt{d}v}_q = \binom{u}{\mathtt{d}v}_q \cdot q^{|\mathtt{d}v|} + \delta_{\mathtt{c},\mathtt{d}} \binom{u}{v}_q .$$

- We choose the recursive definition of q-binomials processed from the left.

Definition (Renard, Rigo & Whiteland 2025)

- $u, v \in \mathcal{A}^*$
- $\mathtt{c}, \mathtt{d} \in \mathcal{A}$

$$\binom{\mathtt{c}u}{\mathtt{d}v}_q = \binom{u}{\mathtt{d}v}_q \cdot q^{|\mathtt{d}v|} + \delta_{\mathtt{c},\mathtt{d}} \binom{u}{v}_q .$$

Example (5)

$$\binom{\mathtt{aab}}{\mathtt{ab}}_q = q^2 + q \quad \text{and} \quad \binom{\mathtt{aba}}{\mathtt{ab}}_q = 1.$$

Theorem (Renard, Rigo & Whiteland 2025)

- $u \in \mathcal{A}^*$
- $k \geqslant 0$
- $\mathtt{a}_1, \ldots, \mathtt{a}_k \in \mathcal{A}$

$$\binom{u}{\mathtt{a}_1 \cdots \mathtt{a}_k}_q = \sum_{\substack{u_1, u_2, \ldots, u_{k+1} \in \mathcal{A}^* \\ u = u_1 \mathtt{a}_1 \cdots u_k \mathtt{a}_k u_{k+1}}} q^{\sum_{i=1}^{k} (k+1-i)|u_i|}.$$

Theorem (Renard, Rigo & Whiteland 2025)

- $u \in \mathcal{A}^*$
- $k \geqslant 0$
- $\mathtt{a}_1, \ldots, \mathtt{a}_k \in \mathcal{A}$

$$\binom{u}{\mathtt{a}_1 \cdots \mathtt{a}_k}_q = \sum_{\substack{u_1, u_2, \ldots, u_{k+1} \in \mathcal{A}^* \\ u = u_1 \mathtt{a}_1 \cdots u_k \mathtt{a}_k u_{k+1}}} q^{\sum_{i=1}^{k} (k+1-i)|u_i|}.$$

- $M = \begin{pmatrix} M_1 & \cdots & M_m \end{pmatrix} \in \mathcal{A}^{**}$
- $P = \begin{pmatrix} P_1 & \cdots & P_p \end{pmatrix} \in \mathcal{A}^{**}$

Theorem (Renard, Rigo & Whiteland 2025)

- $u \in \mathcal{A}^*$
- $k \geqslant 0$
- $\mathtt{a}_1, \ldots, \mathtt{a}_k \in \mathcal{A}$

$$\binom{u}{\mathtt{a}_1 \cdots \mathtt{a}_k}_q = \sum_{\substack{u_1, u_2, \ldots, u_{k+1} \in \mathcal{A}^* \\ u = u_1 \mathtt{a}_1 \cdots u_k \mathtt{a}_k u_{k+1}}} q^{\sum_{i=1}^{k} (k+1-i)|u_i|}.$$

- $M = \begin{pmatrix} M_1 & \cdots & M_m \end{pmatrix} \in \mathcal{A}^{**}$
- $P = \begin{pmatrix} P_1 & \cdots & P_p \end{pmatrix} \in \mathcal{A}^{**}$

Definition (4)

The (q, t)-**column-binomial coefficient** of M and P is

$$\binom{M}{P}_{q,t} = \sum_{1 \leqslant i_1 < \cdots < i_p \leqslant m} t^{\sum_{j=1}^{p} (i_j - j)} \binom{M_{i_1}}{P_1}_q \cdots \binom{M_{i_p}}{P_p}_q.$$

Theorem (Renard, Rigo & Whiteland 2025)

- $u \in \mathcal{A}^*$
- $k \geqslant 0$
- $\mathtt{a}_1, \ldots, \mathtt{a}_k \in \mathcal{A}$

$$\binom{u}{\mathtt{a}_1 \cdots \mathtt{a}_k}_q = \sum_{\substack{u_1, u_2, \ldots, u_{k+1} \in \mathcal{A}^* \\ u = u_1 \mathtt{a}_1 \cdots u_k \mathtt{a}_k u_{k+1}}} q^{\sum_{i=1}^{k} (k+1-i)|u_i|}.$$

- $M = \begin{pmatrix} M_1 & \cdots & M_m \end{pmatrix} \in \mathcal{A}^{**}$
- $P = \begin{pmatrix} P_1 & \cdots & P_p \end{pmatrix} \in \mathcal{A}^{**}$

Definition (4)

The (q,t)-**column-binomial coefficient** of M and P is

$$\binom{M}{P}_{q,t} = \sum_{1 \leqslant i_1 < \cdots < i_p \leqslant m} t^{\sum_{j=1}^{p} (i_j - j)} \binom{M_{i_1}}{P_1}_q \cdots \binom{M_{i_p}}{P_p}_q.$$

Definition (4)

The (q,t)**-column-binomial coefficient** of M and P is

$$\binom{M}{P}_{q,t} = \sum_{1\leqslant i_1<\cdots<i_p\leqslant m} t^{\sum_{j=1}^{p}(i_j-j)} \binom{M_{i_1}}{P_1}_q \cdots \binom{M_{i_p}}{P_p}_q .$$

Definition (4)

The **(q,t)-column-binomial coefficient** of M and P is

$$\binom{M}{P}_{q,t} = \sum_{1\leqslant i_1<\cdots<i_p\leqslant m} t^{\sum_{j=1}^{p}(i_j-j)} \binom{M_{i_1}}{P_1}_q \cdots \binom{M_{i_p}}{P_p}_q .$$

Example (6)

$$M = \begin{pmatrix} \mathtt{a} & \mathtt{a} & \mathtt{a} \\ \mathtt{a} & \mathtt{b} & \mathtt{a} \\ \mathtt{b} & \mathtt{a} & \mathtt{b} \end{pmatrix} \qquad P = \begin{pmatrix} \mathtt{a} & \mathtt{a} \\ \mathtt{b} & \mathtt{b} \end{pmatrix}$$

$$\begin{aligned}\binom{M}{P}_{q,t} &= (q^2+q)\cdot 1 + t\cdot(q^2+q)\cdot(q^2+q) + t^2\cdot 1\cdot(q^2+q)\\ &= q^4t + 2q^3t + q^2t^2 + q^2t + q^2 + qt^2 + q.\end{aligned}$$

1. **Single-row arrays $\implies$ t-specialisation**

$$\binom{M}{P}_{q,t} = \binom{M}{P}_q(t)$$

1. **Single-row arrays $\implies$ t-specialisation**

$$\binom{M}{P}_{q,t} = \binom{M}{P}_q(t)$$

2. **Single-column arrays $\implies$ q-specialisation**

$$\binom{M}{P}_{q,t} = \binom{M}{P}_q$$

Proposition (Renard, Rigo & Whiteland 2025)

- $u, v \in \mathcal{A}^*$
- $\mathtt{a}, \mathtt{b} \in \mathcal{A}$

$$\binom{u\mathtt{a}}{v\mathtt{b}}_q = \binom{u}{v\mathtt{b}}_q \cdot q^{|v\mathtt{b}|} + \delta_{\mathtt{a},\mathtt{b}} \binom{u}{v}_q .$$

Proposition (Renard, Rigo & Whiteland 2025)

- $u, v \in \mathcal{A}^*$
- $\mathtt{a}, \mathtt{b} \in \mathcal{A}$

$$\binom{u\mathtt{a}}{v\mathtt{b}}_q = \binom{u}{v\mathtt{b}}_q \cdot q^{|v\mathtt{b}|} + \delta_{\mathtt{a},\mathtt{b}} \binom{u}{v}_q .$$

Proposition (G. & Rigo 2025)

- $M \in \mathcal{A}^{r\times m}$
- $P \in \mathcal{A}^{s\times p}$
- $C \in \mathcal{A}^{r\times 1}$
- $D \in \mathcal{A}^{s\times 1}$

$$\binom{C \oplus M}{D \oplus P}_{q,t} = \binom{M}{D \oplus P}_{q,t} \cdot t^{p+1} + \binom{C}{D}_q \binom{M}{P}_{q,t} .$$

(q,t)-Chu–Vandermonde's Identity

Proposition (Renard, Rigo & Whiteland 2025)

- $x, y, u \in \mathcal{A}^*$

$$\binom{xy}{u}_q = \sum_{u_1 u_2 = u} q^{|u_1|\left(|y|-|u_2|\right)} \binom{x}{u_1}_q \binom{y}{u_2}_q .$$

Proposition (Renard, Rigo & Whiteland 2025)

- $x, y, u \in \mathcal{A}^*$

$$\binom{xy}{u}_q = \sum_{u_1 u_2 = u} q^{|u_1|\,(|y| - |u_2|)} \binom{x}{u_1}_q \binom{y}{u_2}_q .$$

Proposition (G. & Rigo 2025)

- $M \in \mathcal{A}^{r\times m}$
- $N \in \mathcal{A}^{r\times n}$
- $P \in \mathcal{A}^{s\times p}$
- $|P|$:# columns of P

$$\binom{M \oplus N}{P}_{q,t} = \sum_{P_1 \oplus P_2 = P} t^{(m-|P_1|)|P_2|} \binom{M}{P_1}_{q,t} \binom{N}{P_2}_{q,t} .$$

Symmetry Rules for the (q,t)-Column Binomial

Symmetry Rules for the (q,t)-Column Binomial

- $M = (M_1, \ldots, M_m), \quad P = (P_1, \ldots, P_p), \quad \widetilde{M} = (\text{row-reversed } M), \quad \widetilde{P} = (\text{row-reversed } P).$

- $M = (M_1, \ldots, M_m)$, $P = (P_1, \ldots, P_p)$, $\widetilde{M} = (\text{row-reversed } M)$, $\widetilde{P} = (\text{row-reversed } P)$.

1. **Vertical flip (reverse the column order)**

$$[t^j]\binom{M_1 \cdots M_m}{P_1 \cdots P_p}_{q,t} = [t^{p(m-p)-j}]\binom{M_m \cdots M_1}{P_p \cdots P_1}_{q,t}.$$

   - polynomial in t with coefficients in $\mathbb{N}[q]$

Symmetry Rules for the (q,t)-Column Binomial

- $M = (M_1, \dots, M_m), \quad P = (P_1, \dots, P_p), \quad \widetilde{M} = (\text{row-reversed } M), \quad \widetilde{P} = (\text{row-reversed } P).$

1. **Vertical flip (reverse the column order)**

$$[t^j]\binom{M_1 \cdots M_m}{P_1 \cdots P_p}_{q,t} = [t^{p(m-p)-j}]\binom{M_m \cdots M_1}{P_p \cdots P_1}_{q,t}.$$

   - polynomial in t with coefficients in $\mathbb{N}[q]$

2. **Horizontal flip (reverse each column top–to–bottom)**

$$[q^j]\binom{\widetilde{M}}{\widetilde{P}}_{q,t} = [q^{ps(r-s)-j}]\binom{M}{P}_{q,t}, \quad M \in \mathcal{A}^{r\times m}, P \in \mathcal{A}^{s\times p}.$$

   - polynomial in q with coefficients in $\mathbb{N}[t]$

Example (7)

$$M = \begin{pmatrix} \mathtt{a} & \mathtt{a} & \mathtt{a} \\ \mathtt{a} & \mathtt{b} & \mathtt{a} \\ \mathtt{b} & \mathtt{a} & \mathtt{b} \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} \mathtt{a} & \mathtt{a} \\ \mathtt{b} & \mathtt{b} \end{pmatrix}.$$

Let $r = m = 3$ and $s = p = 2$

$$\binom{M}{P}_{q,t} = q^4 t + 2tq^3 + \left(t^2 + t + 1\right) q^2 + \left(t^2 + 1\right) q,$$

$$\binom{\widetilde{M}}{\widetilde{P}}_{q,t} = \left(t^2 + 1\right) q^3 + \left(t^2 + t + 1\right) q^2 + 2tq + t.$$

(q, t)- Reconstruction Problems

$\boxed{\textbf{The knowledge of certain coefficients uniquely determines the word.}}$

The knowledge of certain coefficients uniquely determines the word.

Example (7)

$$M = \begin{pmatrix} \mathtt{a} & \mathtt{a} & \mathtt{b} \\ \mathtt{a} & \mathtt{b} & \mathtt{b} \\ \mathtt{b} & \mathtt{a} & \mathtt{b} \end{pmatrix} \implies \binom{M}{(\mathtt{b})}_{q,t} = q^2 + tq + t^2\left(q^2 + q + 1\right).$$

$\mathbf{q^{i-1}t^{j-1}}$ **occurs in the** $(\mathbf{q},\mathbf{t})-$**binomial** $\iff$ $\exists\ \mathtt{b} \in \mathcal{A}$ **in position** $(\mathbf{i},\mathbf{j})$

The knowledge of certain coefficients uniquely determines the word.

Example (7)

$$M = \begin{pmatrix} \mathtt{a} & \mathtt{a} & \mathtt{b} \\ \mathtt{a} & \mathtt{b} & \mathtt{b} \\ \mathtt{b} & \mathtt{a} & \mathtt{b} \end{pmatrix} \quad \Longrightarrow \quad \binom{M}{(\mathtt{b})}_{q,t} = q^2 + tq + t^2\left(q^2 + q + 1\right).$$

- Read q^2 $\left(i = 3,\ j = 1\right) \Longrightarrow \mathtt{b}$ in row 3, column 1.
- Read tq $\left(i = 2,\ j = 2\right) \Longrightarrow \mathtt{b}$ in row 2, column 2.
- Read t^2q^2 $\left(i = 3,\ j = 3\right) \Longrightarrow$ etc.

The knowledge of certain coefficients uniquely determines the word.

Example (7)

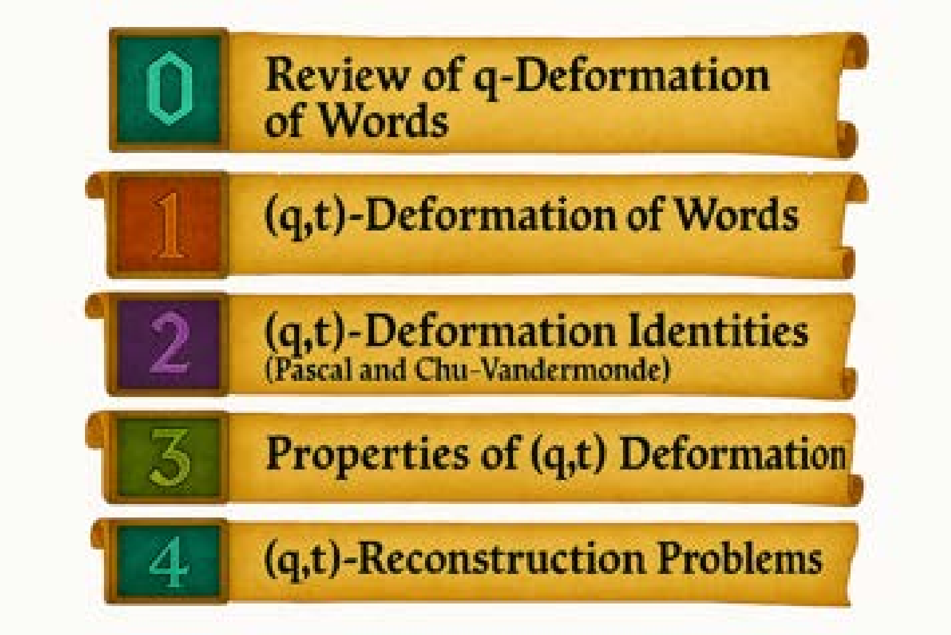
$$M = \begin{pmatrix} \mathtt{a} & \mathtt{a} & \mathtt{b} \\ \mathtt{a} & \mathtt{b} & \mathtt{b} \\ \mathtt{b} & \mathtt{a} & \mathtt{b} \end{pmatrix} \implies \binom{M}{(\mathtt{b})}_{q,t} = q^2 + tq + t^2\left(q^2 + q + 1\right).$$

- Read q^2 $\left(i = 3,\ j = 1\right) \implies \mathtt{b}$ in row 3, column 1.
- Read tq $\left(i = 2,\ j = 2\right) \implies \mathtt{b}$ in row 2, column 2.
- Read t^2q^2 $\left(i = 3,\ j = 3\right) \implies$ etc.

$$\boxed{\mathbf{q^{i-1}t^{j-1}} \textbf{occurs in the} (\mathbf{q},\mathbf{t}) - \textbf{binomial} \iff \exists\, \mathtt{b} \in \mathcal{A} \textbf{ in position} (\mathbf{i},\mathbf{j})}$$

Recap(II)

Recap(II)



0
Review of q-Deformation of Words
1
(q,t)-Deformation of Words
2
(q,t)-Deformation Identities
(Pascal and Chu-Vandermonde)
3
Properties of (q,t) Deformation
4
(q,t)-Reconstruction Problems

WORDS 2025, Nancy (!)

Figure: Villa Majorelle