Circularity and repetitiveness in non-injective D0L systems

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Part 1

DF0L systems

Definition

P	**D**	**F**	**0**	**L**
Propagating	Deterministic	Finite-axiom	Zero-context	Lindenmayer

$$S = (A, \varphi\colon A^* \to A^+, W \subseteq A^+)$$
$$\mathcal{L}(S) = \{u \in A^* \mid u \leq_{\text{fac}} \varphi^n(w) \text{ for some } n \in \mathbb{N} \text{ and } w \in W\}.$$

- A is a finite **alphabet**.
- $\varphi\colon A^* \to A^*$ is a **morphism**.
- $W \subseteq A^+$ is a finite set of words called **axioms**.

The notion of DF0L system is closely related to the notion of **substitutive shifts**, also called **purely morphic shifts**.

Some examples

$$S = (\{a, b\},\ \varphi\colon a \mapsto ab, b \mapsto ba,\ a).$$

Axiom images:

$a, ab, abba, abbabaab, abbabaabbaababba, \ldots$

Language:

$\{\varepsilon, a, b, aa, ab, ba, bb, aab, aba, abb, baa, bab, bba, \ldots\}$

Some examples

$$S = (\{a, b\},\ \varphi\colon a \mapsto abba, b \mapsto bab,\ aa).$$

Axiom images:

$aa, abbaabba, abbababbababbaabbababbababba, \ldots$

Language:

$\{\varepsilon, a, b, ab, ba, aa, bb, baa, bab, abb, bba, aba, aab, \ldots\}$

Some examples

$$S = (\{a, b, c, d\},\ \varphi\colon a \mapsto baa, b \mapsto adc, c \mapsto cdc, d \mapsto ad,\ c).$$

Axiom images:

$c, cdc, cdcadcdc, cdcadcdcbaaadcdcadcdc, \ldots$

Language:

$\{\varepsilon, a, b, c, d, aa, ab, ad, ba, ca, cb, cd, dc, aaa, aab, aad, aba, \ldots\}$

Some examples

$$S = (\{a, b, c, d\},\ \varphi\colon a \mapsto cad, b \mapsto cbd, c \mapsto c, d \mapsto c,\ \{a, b\}).$$

Axiom images:

$a, b, cad, cbd, ccadd, ccbdd, cccaddd, cccbddd, \ldots$

Language:

$\{\varepsilon, a, b, c, d, ad, bd, ca, cb, cc, dd, add, bdd, cad, cbd, cca, ccb, \ldots\}$

Some examples

$$S = (\{a, b, c\},\ \varphi\colon a \mapsto abaca, b \mapsto aba, c \mapsto aba,\ a).$$

Axiom images:

$a, abaca, abacaabaabacaabaabaca, \ldots$

Language:

$\{\varepsilon, a, b, c, aa, ab, ac, ba, ca, aab, aba, aca, baa, bac, caa, \ldots\}$

Definition 2.3

A D0L system is **eventually injective** if the following set is finite:

$$\Delta_S = \{\{u, v\} \subseteq \mathcal{L}(S) \mid u \neq v, \varphi(u) = \varphi(v)\}.$$

It is called injective if moreover $\Delta_S = \varnothing$.

In a PD0L system we can measure failure of injectivity with the quantity

$$\delta_S = \max_{\{u,v\} \in \Delta_S} \{|\varphi(u)|\}.$$

The system S is eventually injective if and only if $\delta_S < \infty$ and injective if and only if $\delta_S = 0$ ($\max \varnothing = 0$).

The requirement that $\{u, v\} \subseteq \mathcal{L}(S)$ makes eventual injectivity difficult to decide.

Example 2.4

Let $S = (\{a, b, c\},\ \varphi\colon a \mapsto abacc, b \mapsto aba, c \mapsto aba,\ a)$.

The system S is eventually injective with

$$\Delta_S = \{\{b, c\}, \{ba, ca\}, \{ab, ac\}, \{bac, cab\}\}, \quad \delta_S = 11.$$

Example 2.5

Let $S = (\{a, b, c\},\ \varphi\colon a \mapsto abaca, b \mapsto aba, c \mapsto aba,\ a)$.

The system S is not eventually injective. An infinite sequence of elements $\{u_n, v_n\} \in \Delta_S$ can be constructed inductively by

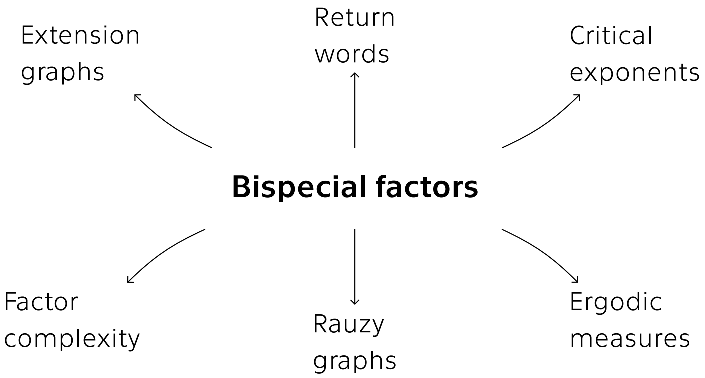
$$(u_1, v_1) = (aca, aba), \quad (u_{n+1}, v_{n+1}) = (u_1\varphi(u_n), v_1\varphi(v_n)).$$

Part 2

Main result

Motivations

The notions of weak and strong circularity are central in the construction of a data structure describing **bispecial factors** (Klouda, 2012). In this context, effective calculation of the thresholds is essential.



Some history

Mignosi & Séébold, 1993	Showed equivalence between strong circularity and unbounded repetitiveness.
Cassaigne, 1994	Introduced weak circularity and used it to study pattern-avoidance in HD0L systems.
Klouda, 2012	Used weak and strong circularity to describe bispecial factors in PD0L systems.
Klouda & Starosta, 2019	Revisited Mignosi and Séébold's result, fixing gaps in the proof.

In all of those papers, injectivity is assumed for most results. Our main goal is to explore what happens when the injectivity assumption is weakened or dropped entirely.

Repetitiveness

Definition

A D0L system S is called **repetitive** if there is a word $u \in A^+$ such that $u^n \in \mathcal{L}(S)$ for all $n \in \mathbb{N}$. We say that u is a **repetition**.

The system S is called **unboundedly repetitive** if there is a repetition u which is unbounded in the following sense:

$$\lim_{n \to \infty} |\varphi^n(u)| < \infty.$$

Unbounded repetitiveness is a decidable property (Ehrenfeucht & Rozenberg, 1983; Mignosi & Séébold, 1993; Klouda & Starosta, 2015).

Main result

Our main result relates unbounded repetitiveness with two conditions called **weak circularity** and **strong circularity**.

Theorem

Let S be a PD0L system.

1. If S is **not weakly circular** then it is **unboundedly repetitive**.
2. If S is **unboundedly repetitive** then it is **not strongly circular**.

- Strong circularity implies weak circularity.
- The converse fails in general.
- The converse holds for eventually injective systems.

Main result in diagram

$$
\begin{tikzcd}
& \text{Not unboundedly repetitive} \arrow[dl, Rightarrow, shift right, gray, "?"'] \arrow[dr, Rightarrow, shift right] & \\
\text{Strongly circular} \arrow[ur, Rightarrow, shift right] \arrow[rr, Rightarrow] & & \text{Weakly circular} \arrow[ul, Rightarrow, shift right, red, "/" marking] \arrow[ll, Rightarrow, bend left, yellow]
\end{tikzcd}
$$

- The negated red implication indicates a counter-example.
- The yellow implication holds in the eventually injective case.
- The gray implication with a question mark remains open.

Taking powers

Let $S = (A, \varphi, W)$ be a DF0L system. Define the nth power of S by

$$S^n = (A, \varphi^n, \{\varphi^i(w) \mid i < n, w \in W\}).$$

This system satisfies $\mathcal{L}(S^n) = \mathcal{L}(S)$. The extra axioms are important.

If S^n is weakly circular for some $n \geq 1$, then so is S (Proposition 3.12). However, the converse is false in general (Example 3.13).

Corollary 4.3

Let S be a strongly circular PDF0L system. Then S^n is weakly circular for all $n \geq 1$.

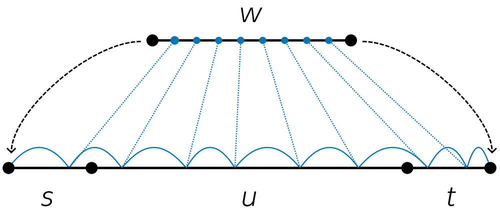
We do not know if S^n is in fact always strongly circular.

Part 3

Circularity

Definition 3.1

Let $S = (A, \varphi, W)$ be a DF0L system. An **interpretation** of $u \in \mathcal{L}(S)$ is a triple (s, w, t) such that $\varphi(w) = sut$.



An interpretation is **minimal** if it has no proper factor which is also an interpretation. If (s, w, t) is a minimal interpretation of u,

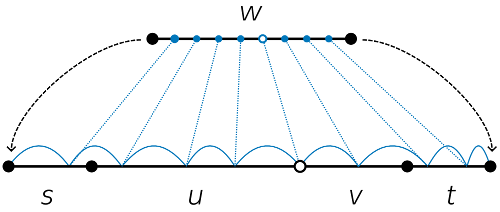
$$|w| \leq 2 + (|u| - 2)/\min_{a \in A}\{|\varphi(a)|\}.$$

Admissible pair

Definition

Let $S = (A, \varphi, W)$ be a D0L system. A pair (u, v) is said to be **compatible** with an interpretation (s, w, t) if there is a factorization $w = w_1 w_2$ such that $\varphi(w_1) = u$ and $\varphi(w_2) = v$.

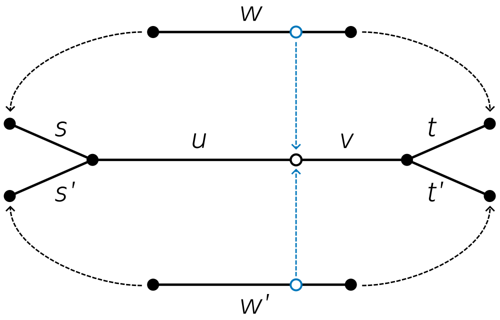
A pair is called **admissible** if it is compatible with at least one interpretation.



Weakly synchronizing pair

Definition 3.4

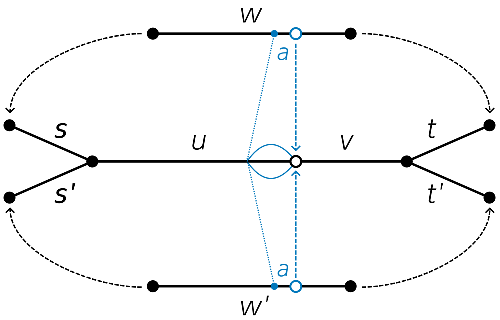
Let $S = (A, \varphi, W)$ be a DF0L system. A pair (u, v) is said to be **weakly synchronizing** if it is compatible with every interpretation of uv.



Strongly synchronizing pair

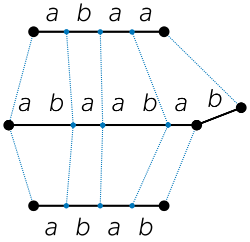
Definition 3.4

Let $S = (A, \varphi, W)$ be a DF0L system. A pair (u, v) is said to be **strongly synchronizing** if it is compatible with every interpretation of uv and for every interpretation (s, w, t) of uv the last letter of w_1 such that $\varphi(w_1) = u$ is constant.



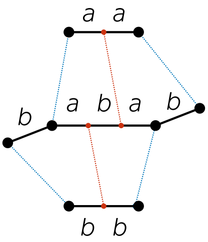
Some examples

$$S = (\{a, b\},\ \varphi\colon a \mapsto ab, b \mapsto a,\ a)$$



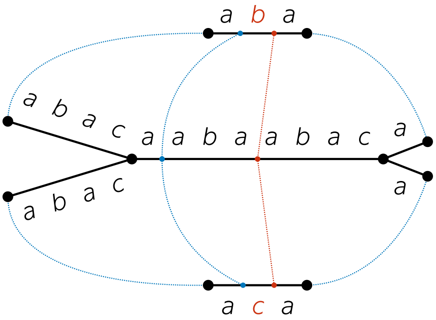
Some examples

$$S = (\{a, b\},\ \varphi\colon a \mapsto ab, b \mapsto ba,\ a)$$

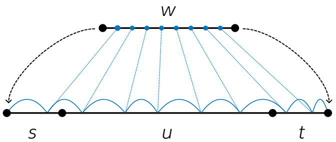


Some examples

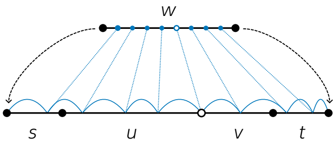
$$S = (\{a, b, c\},\ \varphi\colon a \mapsto abaca, b \mapsto aba, c \mapsto aba,\ a)$$



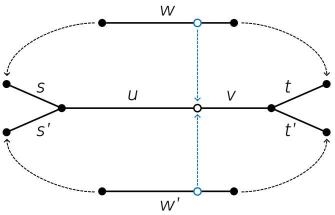
Recap



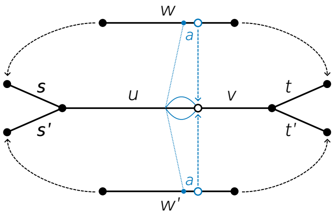
Interpretation



Admissible pair



Weakly synchronizing pair



Strongly synchronizing pair

Weak and strong circularity

Definition 3.7

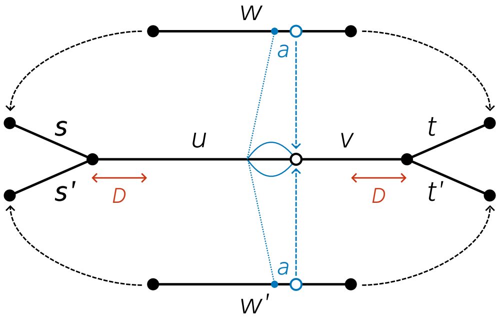
Let $S = (A, \varphi, w)$ be a DF0L system.

1. We say that S is **weakly circular** if there exists $D \geq 0$ such that all $u \in \mathcal{L}(S)$ with $|u| > D$ admit a synchronizing pair.
2. We say that S is **strongly circular** if there exists $D' \geq 0$ such that all admissible pairs (u, v) with $|u|, |v| > D'$ is strongly synchronizing.

The smallest values for D and D', denoted D_{w} and D_{s}, are called the weak and strong **circularity thresholds**.

Strong circularity delay

In strong circularity, the constant D_s represents the delay before admissible pairs become strongly synchronizing.



Finding the thresholds

Lemma 3.6

Let S be a DF0L system. Let (u_1, u_2) and (v_1, v_2) be two pairs such that $v_1 \leq_s u_1$ and $v_2 \leq_p u_2$. If (v_1, v_2) is weakly or strongly synchronizing, then so is (u_1, u_2).

Therefore, we can work only with **minimal interpretations**. In particular, we can effectively test whether a given integer $D \geq 0$ satisfies $D_w \leq D$ or $D_s \leq D$.

If a DF0L system is known to be weakly or strongly circular then we can calculate D_w and D_s by testing successive values of $D \geq 0$.

This procedure can be made reasonably efficient by reusing information from smaller values of D.

Weak and strong thresholds

Proposition 3.11

Let $S = (A, \varphi, W)$ be a PDF0L system.

1. If S is strongly circular then it is weakly circular with

$$D_{\mathrm{w}} \leq 2D_{\mathrm{s}} + \max_{a \in A}\{|\varphi(a)|\}.$$

2. If S is weakly circular and eventually injective then it is strongly circular with

$$D_{\mathrm{s}} \leq D_{\mathrm{w}} + \delta_S + 1.$$

Example 3.13

Let $S = (\{a, b, c\},\ \varphi\colon a \mapsto aac, b \mapsto bc, c \mapsto bc, a)$. Then S is weakly circular with $D_{\mathrm{w}} = 1$, but S^2 is not weakly circular. Thus, S is not strongly circular.

Examples of circularity thresholds (1/2)

Example

$S = (\{a, b\},\ \varphi\colon a \mapsto ab, b \mapsto a,\ a)$. $D_{\mathrm{w}} = D_{\mathrm{s}} = 0$.

Example

$S = (\{a, b\},\ \varphi\colon a \mapsto ba, b \mapsto a,\ a)$. $D_{\mathrm{w}} = 0$ and $D_{\mathrm{s}} = 1$.

Example

$S = (\{a, b\},\ \varphi\colon a \mapsto ab, b \mapsto ba,\ a)$. $D_{\mathrm{w}} = 3$ and $D_{\mathrm{s}} = 1$.

Example

$S = (\{a, b\},\ \varphi\colon a \mapsto abba, b \mapsto bab,\ aa)$. $D_{\mathrm{w}} = 15$ and $D_{\mathrm{s}} = 7$.

Examples of circularity thresholds (2/2)

Example

$S = (\{a, b, c, d\},\ \varphi\colon a \mapsto cad, b \mapsto cbd, c \mapsto c, d \mapsto c,\ \{a, b\})$. $D_{\mathrm{w}} = D_{\mathrm{s}} = 1$.

Example

$S = (\{a, b, c, d\},\ \varphi\colon a \mapsto cb, b \mapsto ad, c \mapsto c, d \mapsto c,\ \{a, b\})$. $D_{\mathrm{w}} = 0$ and $D_{\mathrm{s}} = 1$.

Example

$S = (\{a, b, c\},\ \varphi\colon a \mapsto abacc, b \mapsto aba, c \mapsto aba,\ a)$. $D_{\mathrm{w}} = D_{\mathrm{s}} = 3$.

Example

$S = (\{a, b, c\},\ \varphi\colon a \mapsto abaca, b \mapsto aba, c \mapsto aba,\ a)$. $D_{\mathrm{w}} = 3$ and $D_{\mathrm{s}} = 9$.

D0L online

We created an online tool to analyze bispecial factors in DF0L systems called **D0L online**. It is not operational yet, but we are working on it!

https://d0l.kam.fit.cvut.cz/