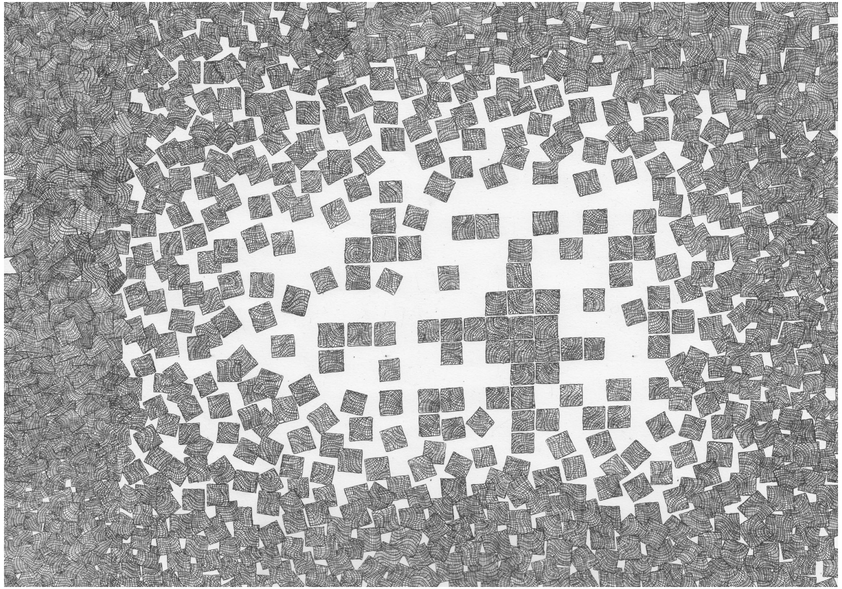
Shuffle squares and nest-free graphs



Jarosław Grytczuk
Warsaw University of Technology

Bartłomiej Pawlik
Silesian University of Technology

Andrzej Ruciński
Adam Mickiewicz University

Tangrams, matchings, and shuffle squares

A B B A B A B A A A

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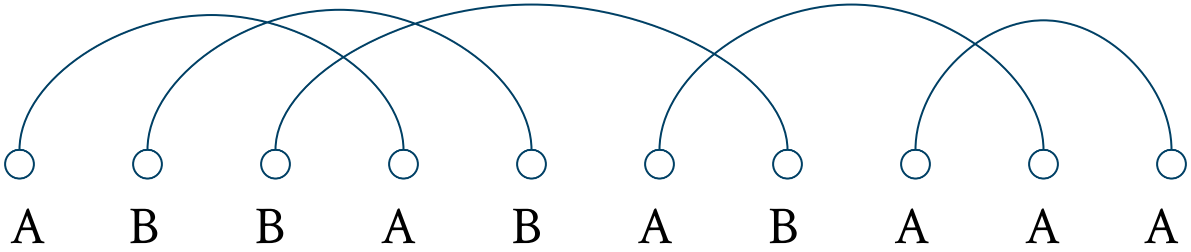

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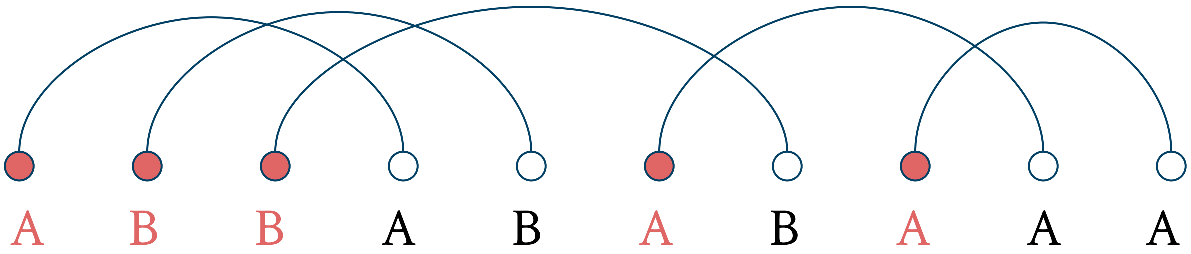

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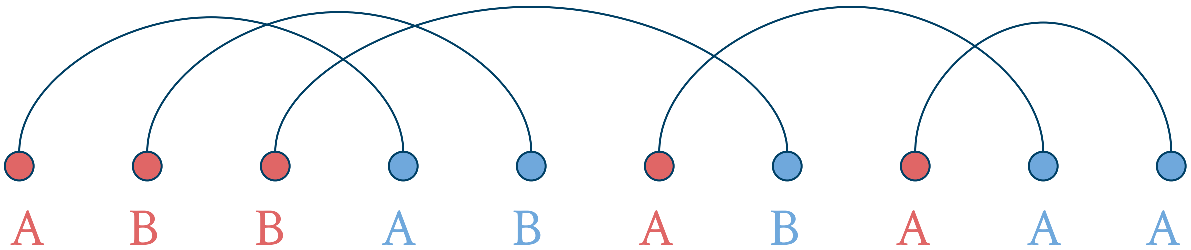

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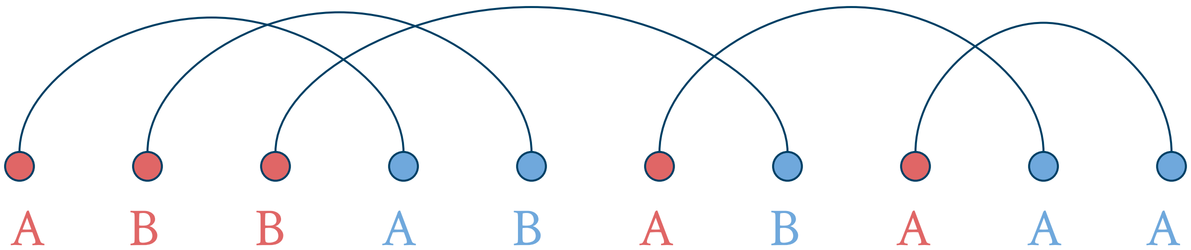

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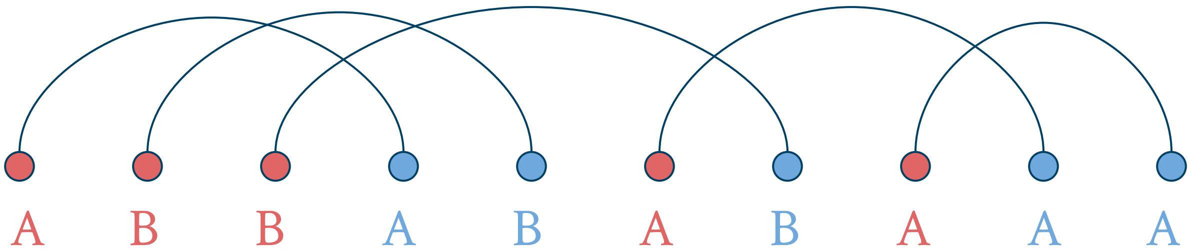

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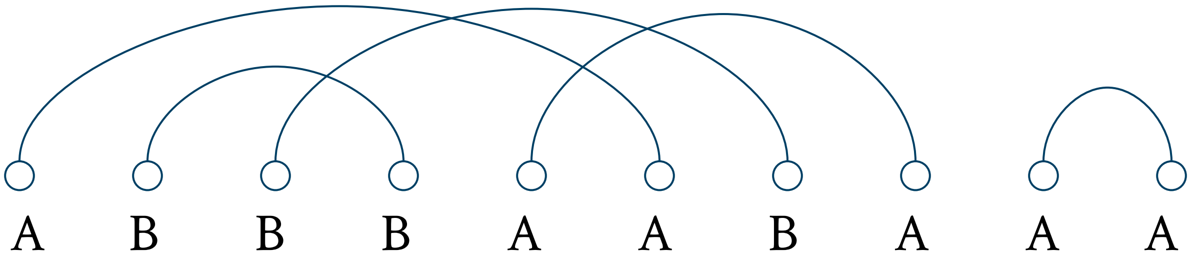

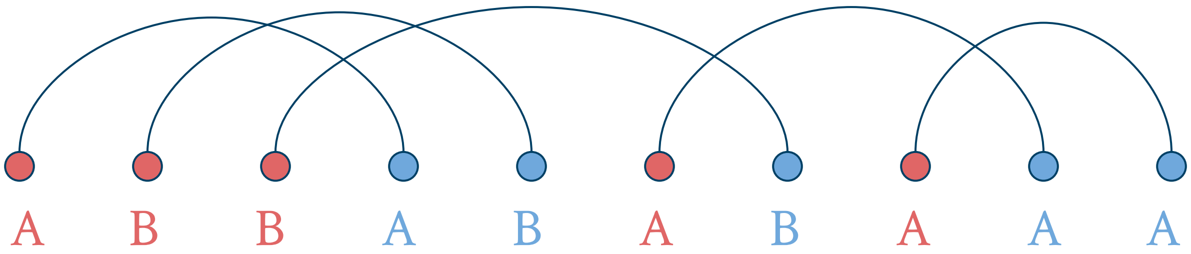

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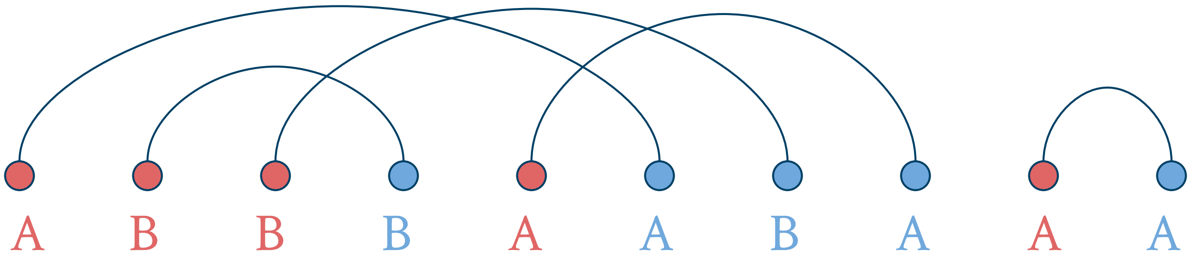

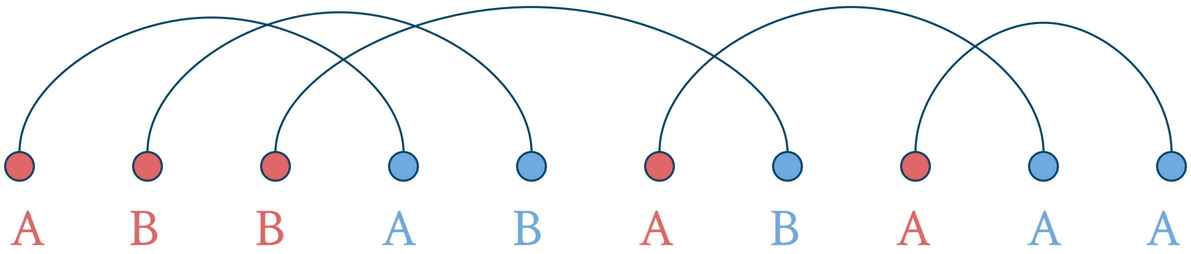

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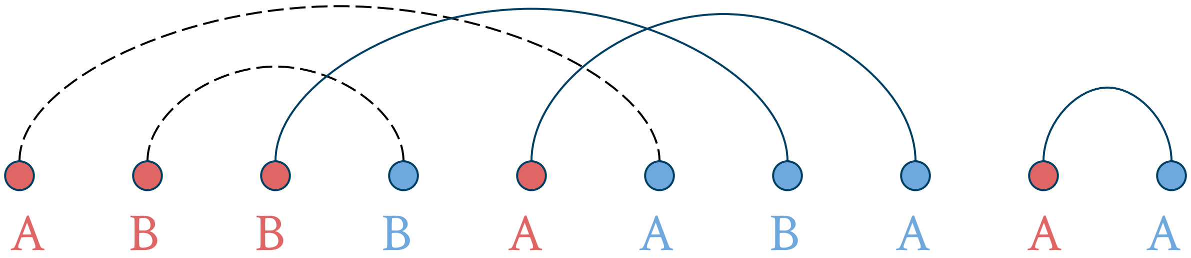

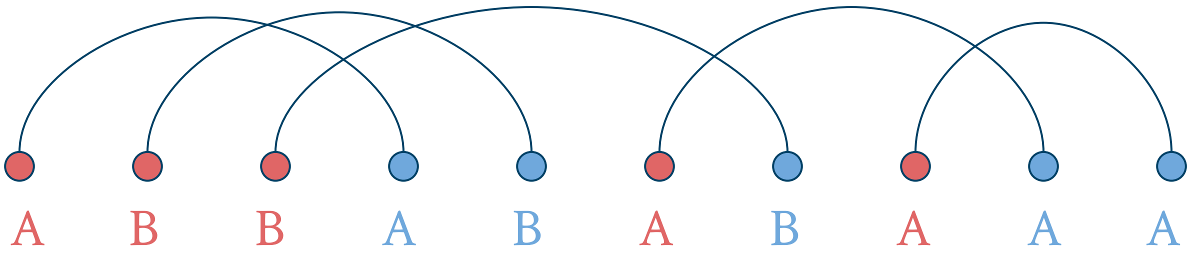

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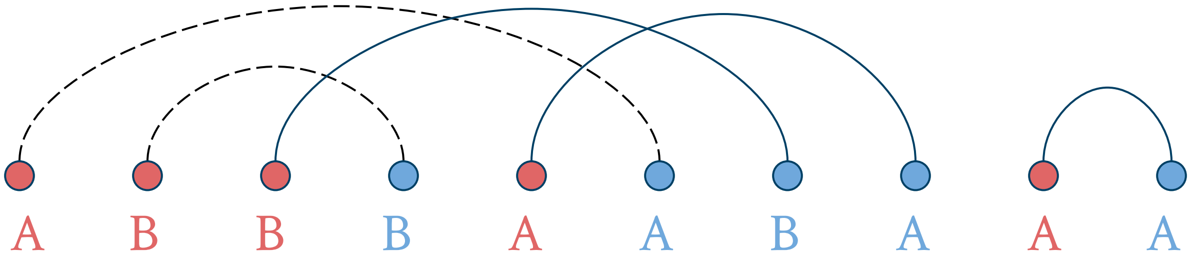

Fact: A word is a *shuffle square* if and only if it has a *perfect matching* without a *nesting*.

Problem (Henshall, Rampersad, Shallit, 2012):

Determine a simple closed form for the number of *binary* shuffle squares of length $2n$.

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The odd ABBA problem

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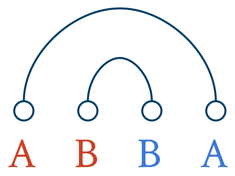

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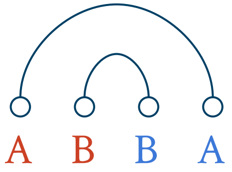

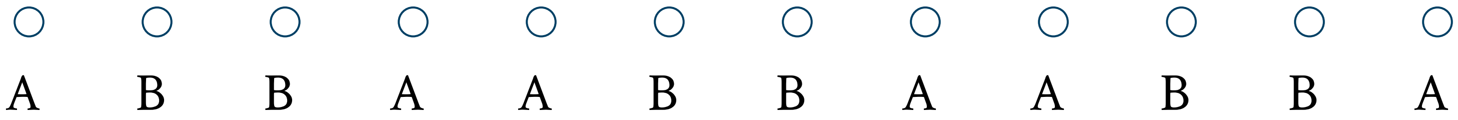

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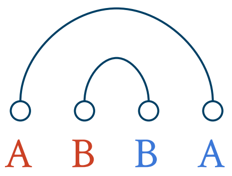

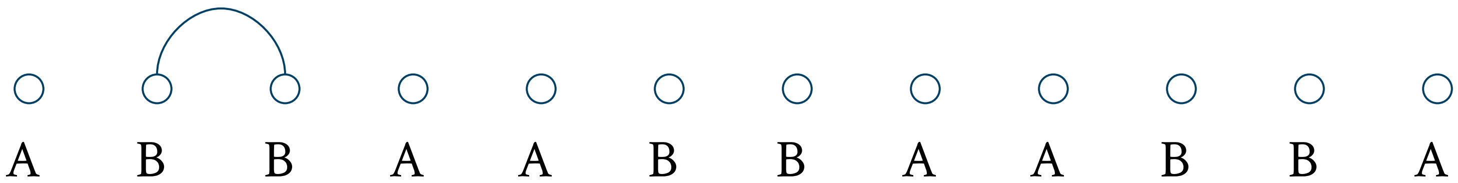

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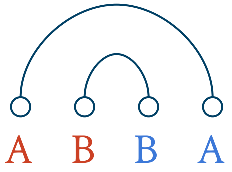

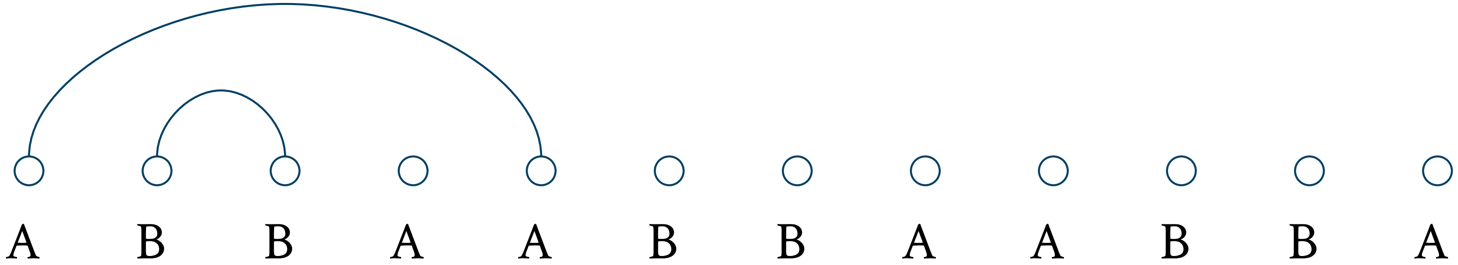

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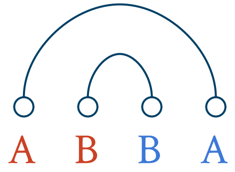

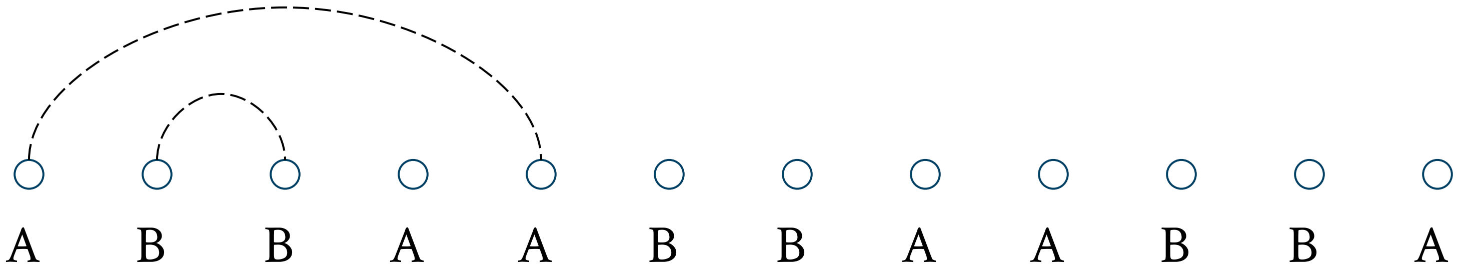

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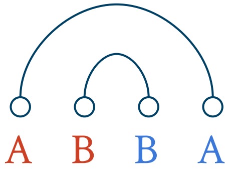

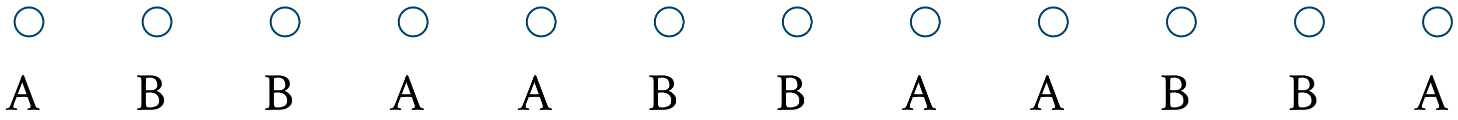

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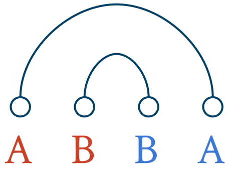

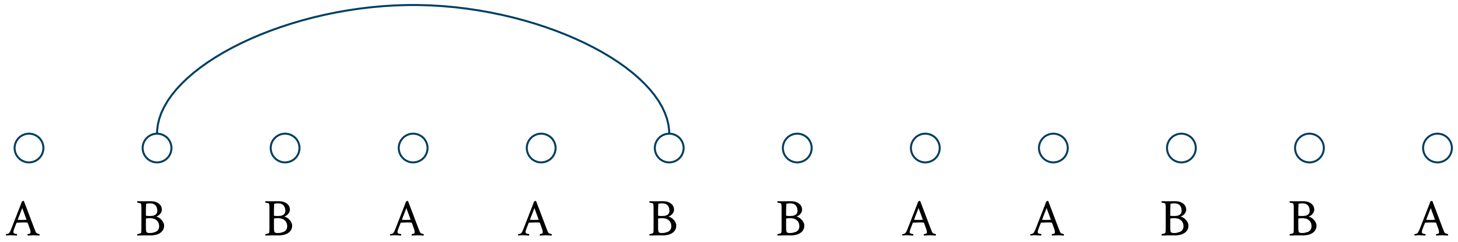

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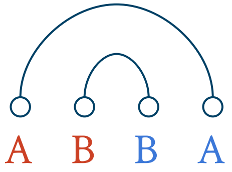

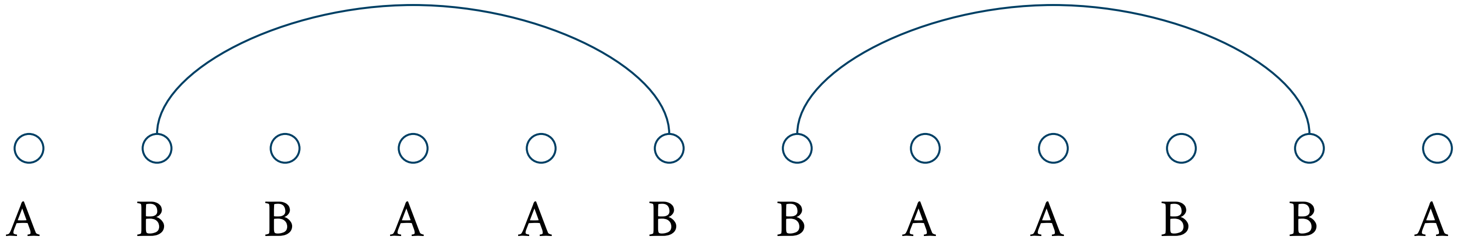

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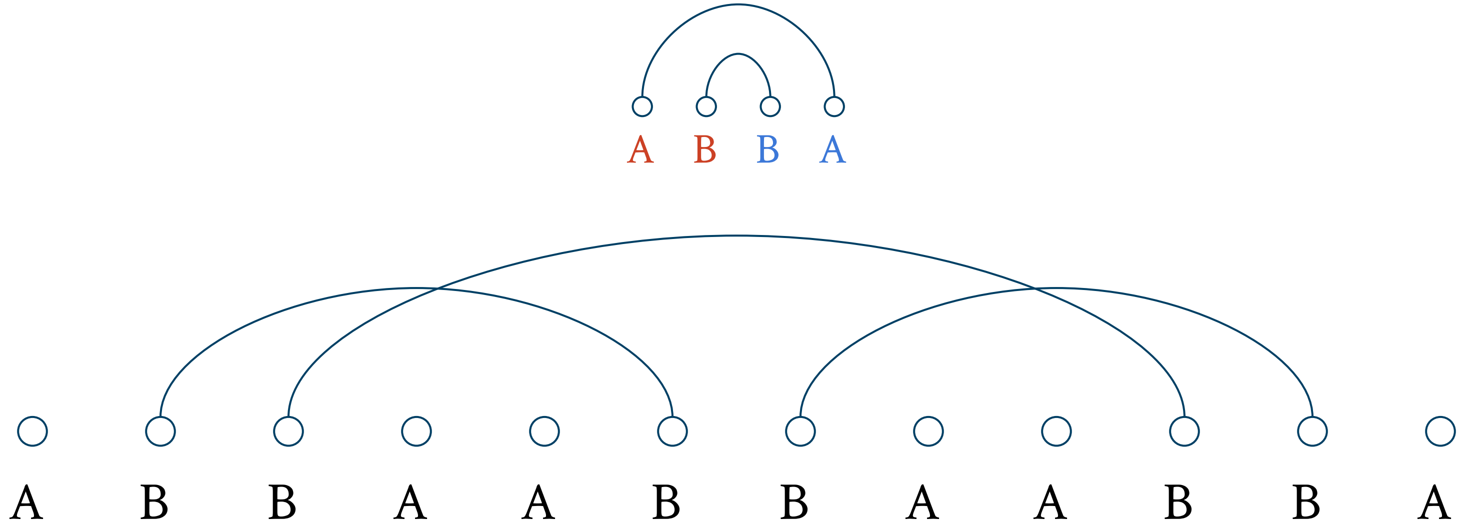

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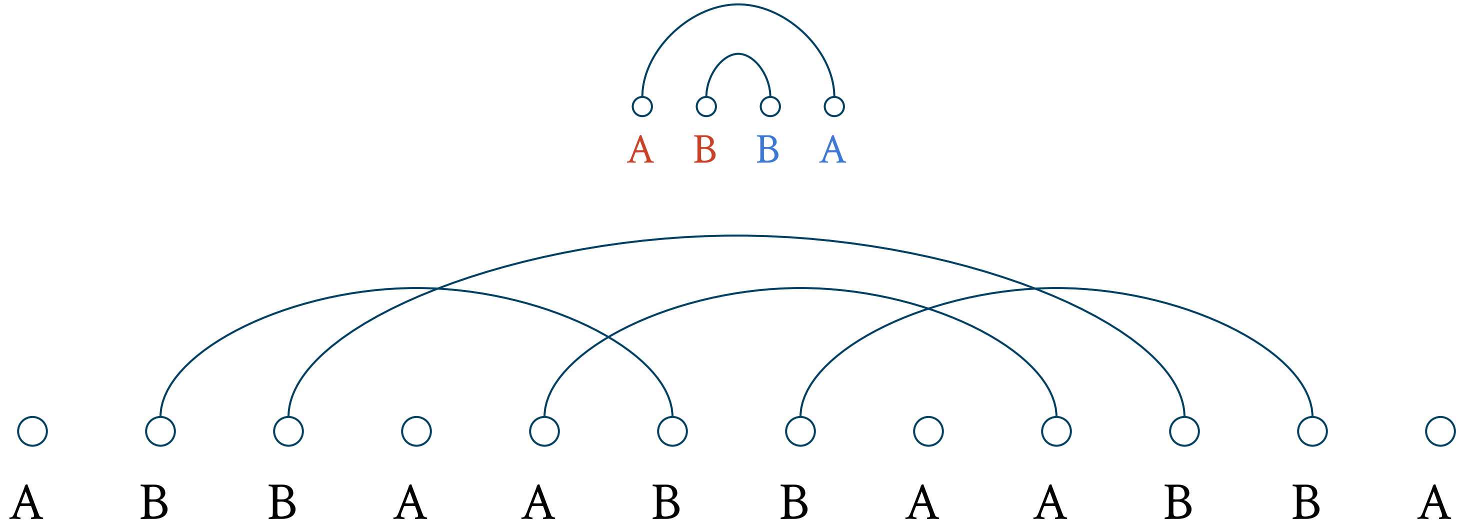

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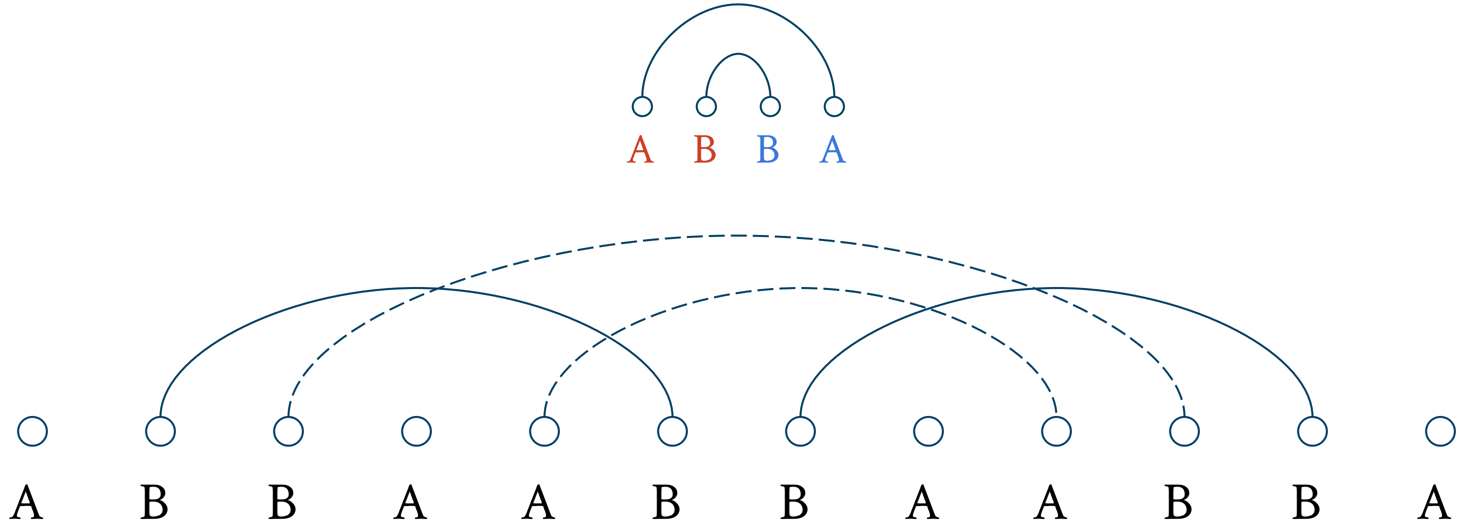

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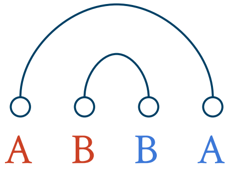

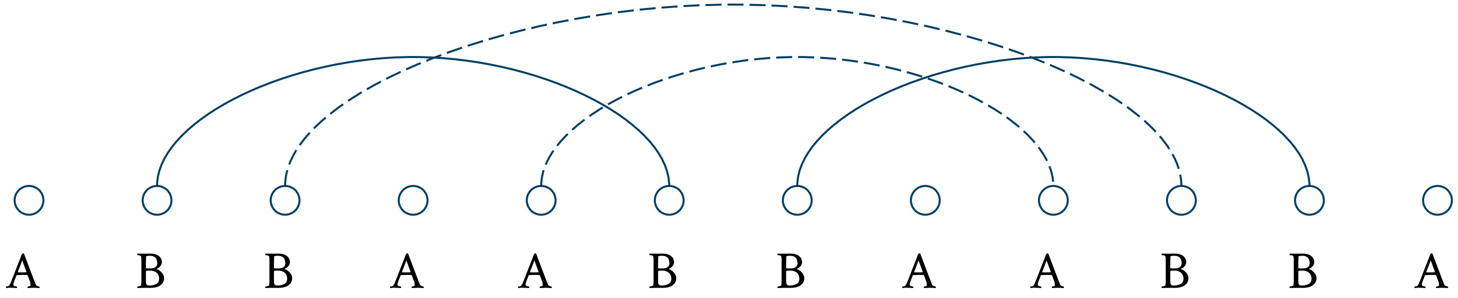

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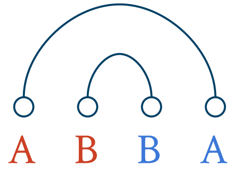

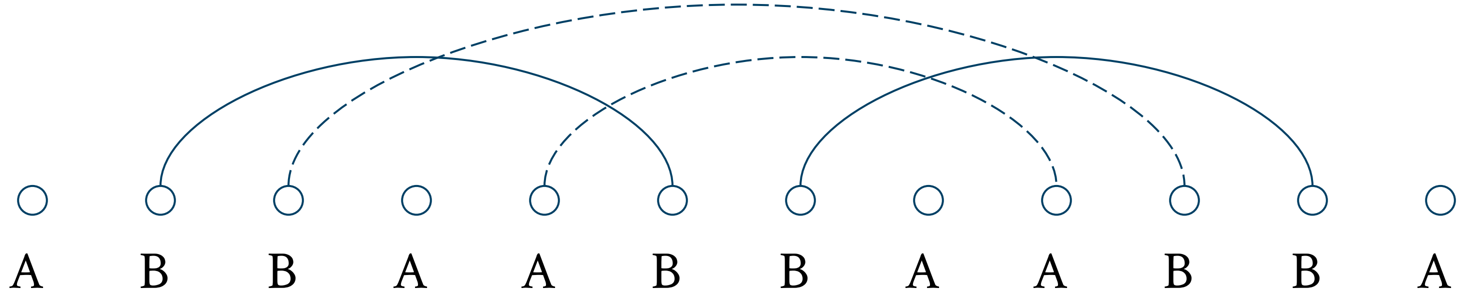

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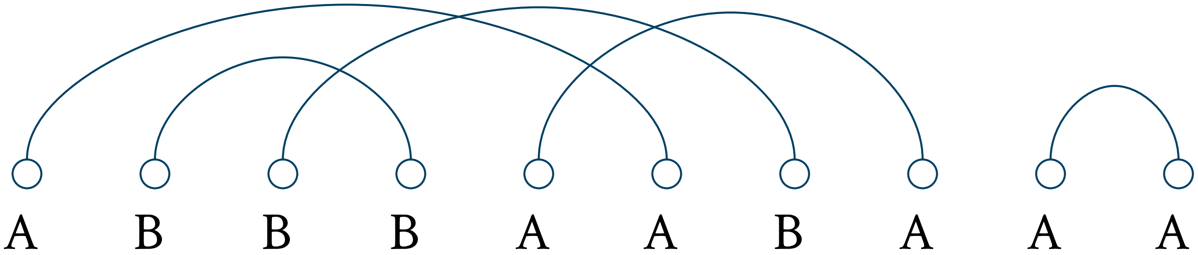
Proof not verified and not published.

The lemma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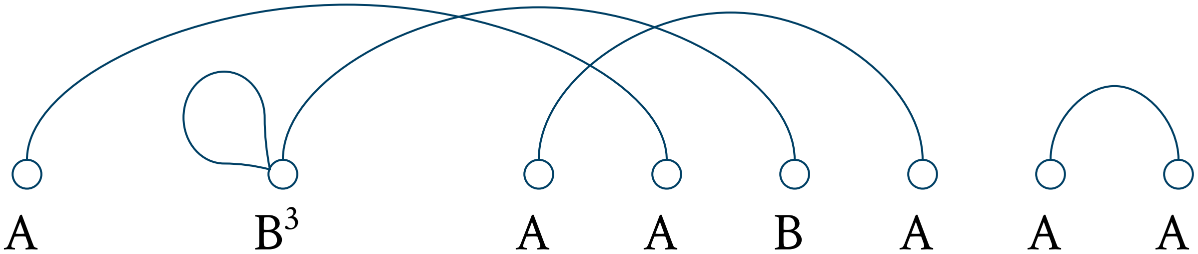
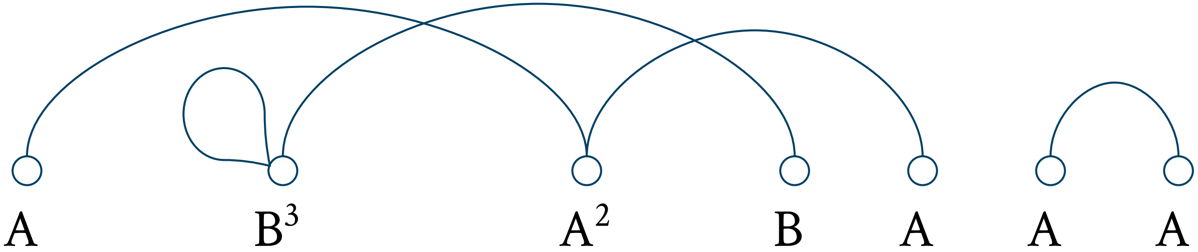
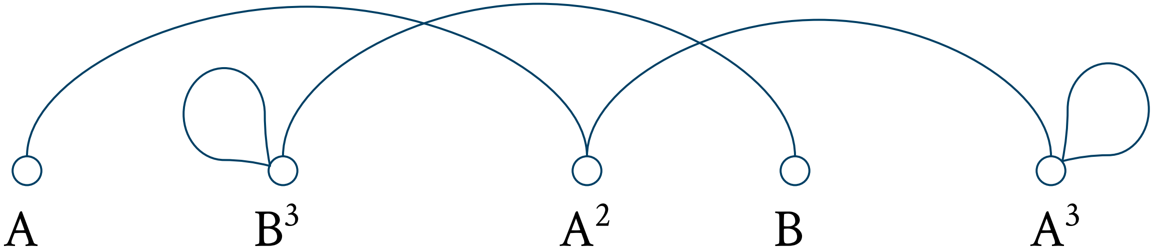

A
B^3
A
A
B
A
A
A


A
B3
A2
B
A
A
A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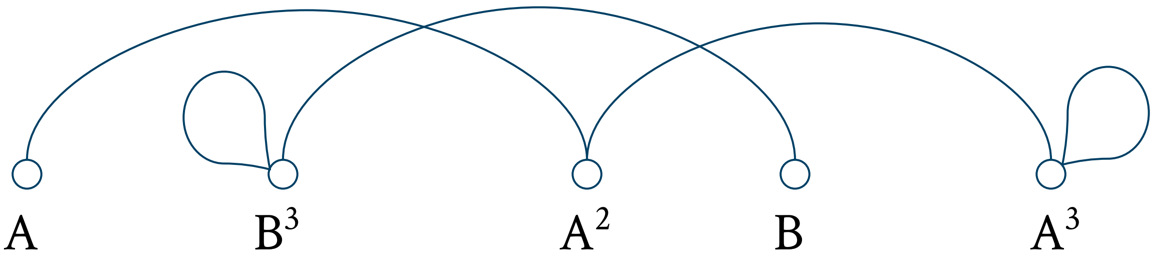
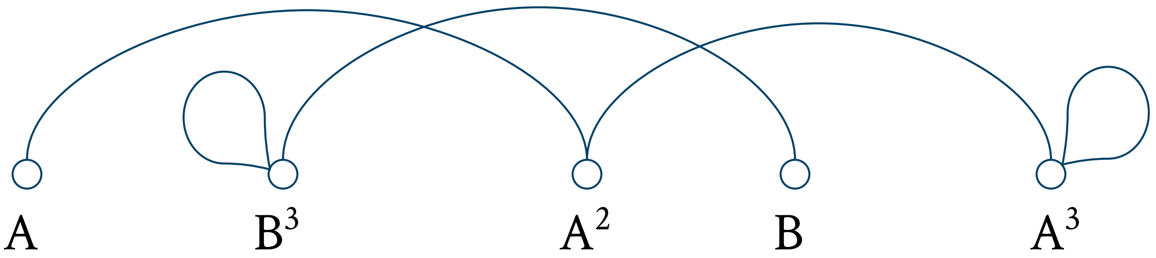
B^3
A^2
B
A^3



Definition: Let $W = R(1)R(2)\dots R(n)$ be a word, where $R(i)$ are consecutive *runs* of letters. A *graph* of W is any *ordered graph* $G(W)$ on the vertices $\{1, 2, \dots, n\}$, with *degrees* specified by the lengths of the runs $R(i)$, that is, $\deg(i) = |R(i)|$.



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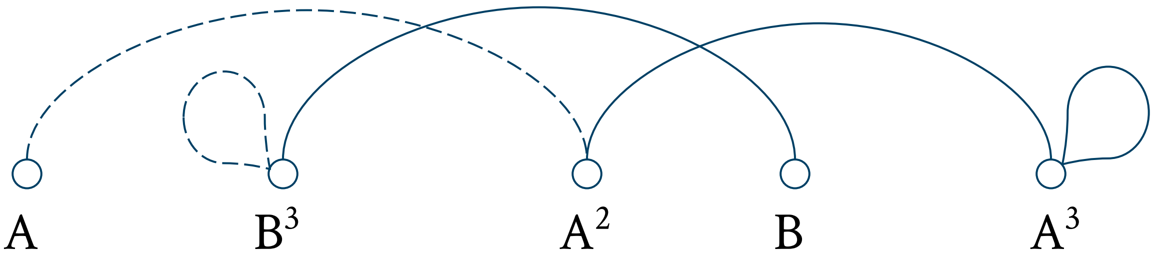
Fact: A word W is a *shuffle square* if and only if it has a graph $G(W)$ with no *nesting*.



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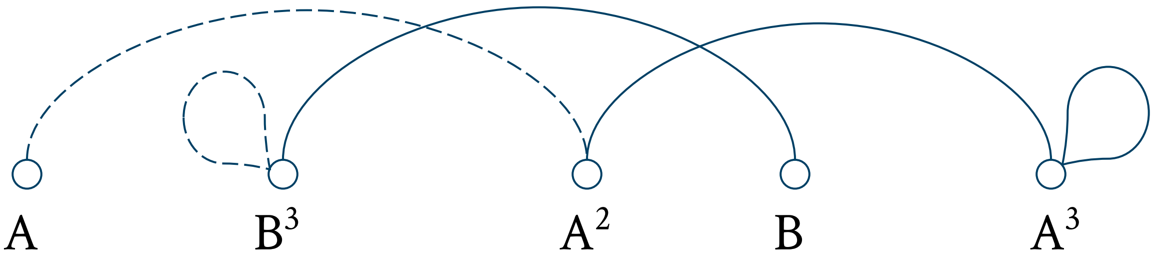

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Let G be an *ordered graph* on the vertices $\{1, 2, \dots, n\}$ consisting of two *disjoint components*, a *path* P starting in 1 and ending in n, and a *cycle* C. If G does not contain a *nesting*, then C must be a cycle with *even* number of vertices.

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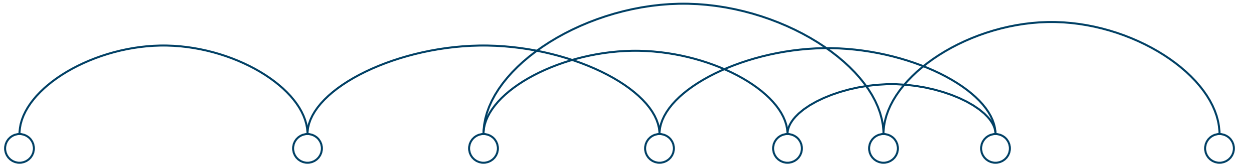

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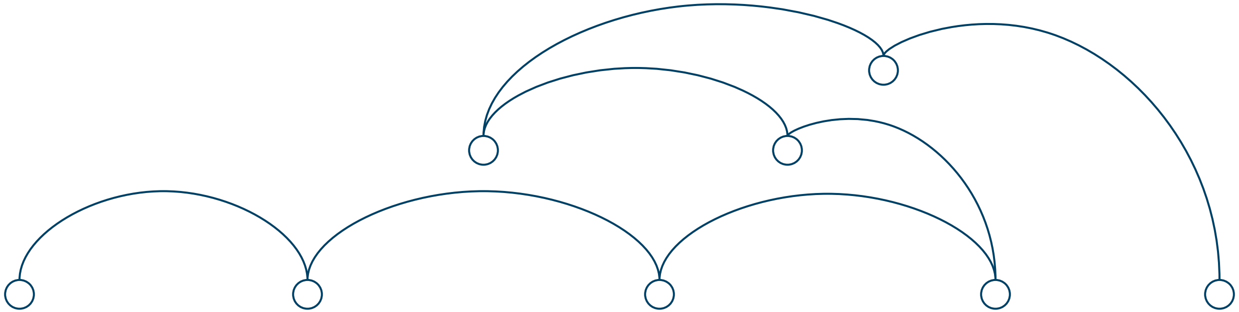

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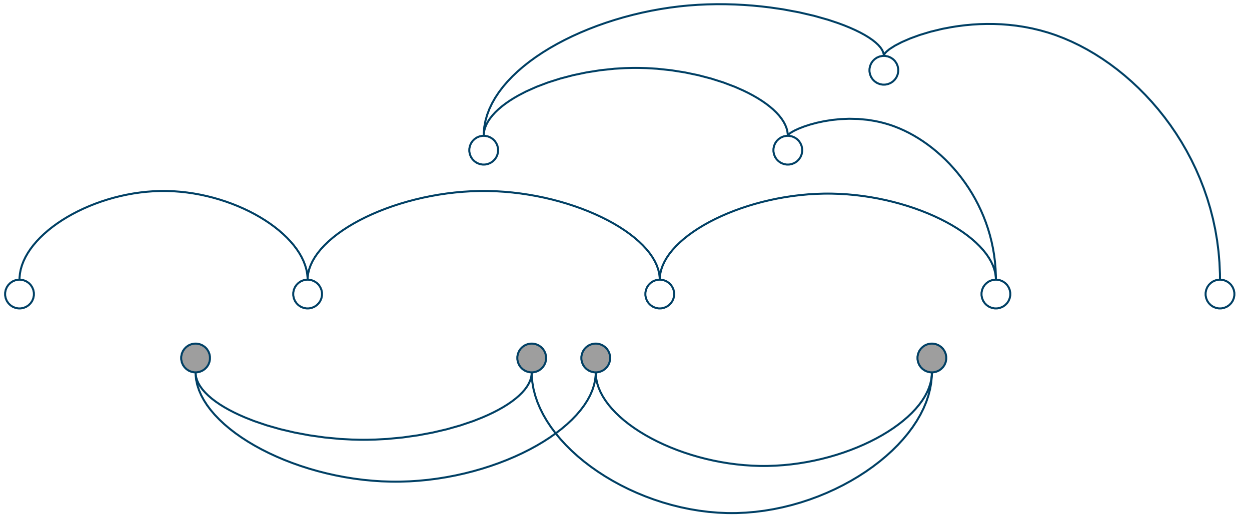

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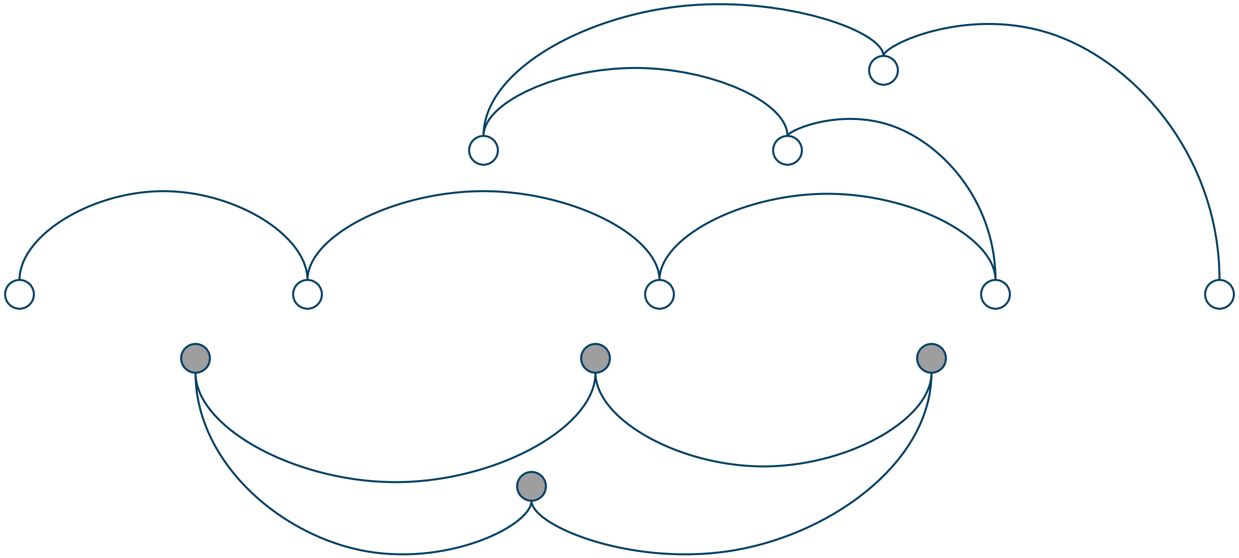

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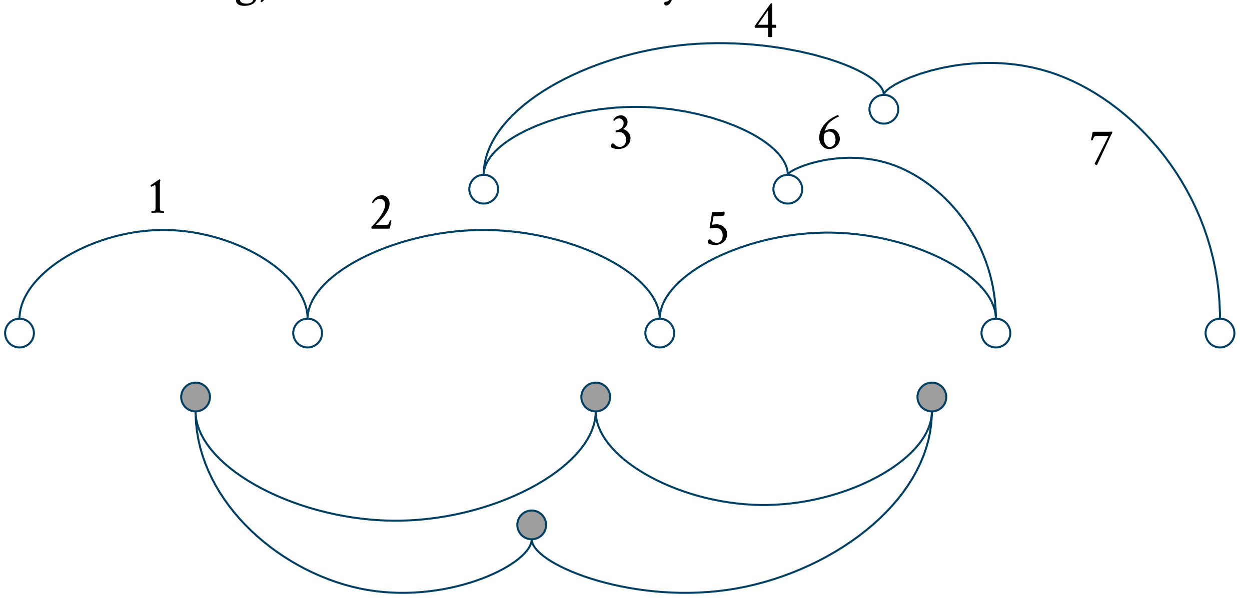

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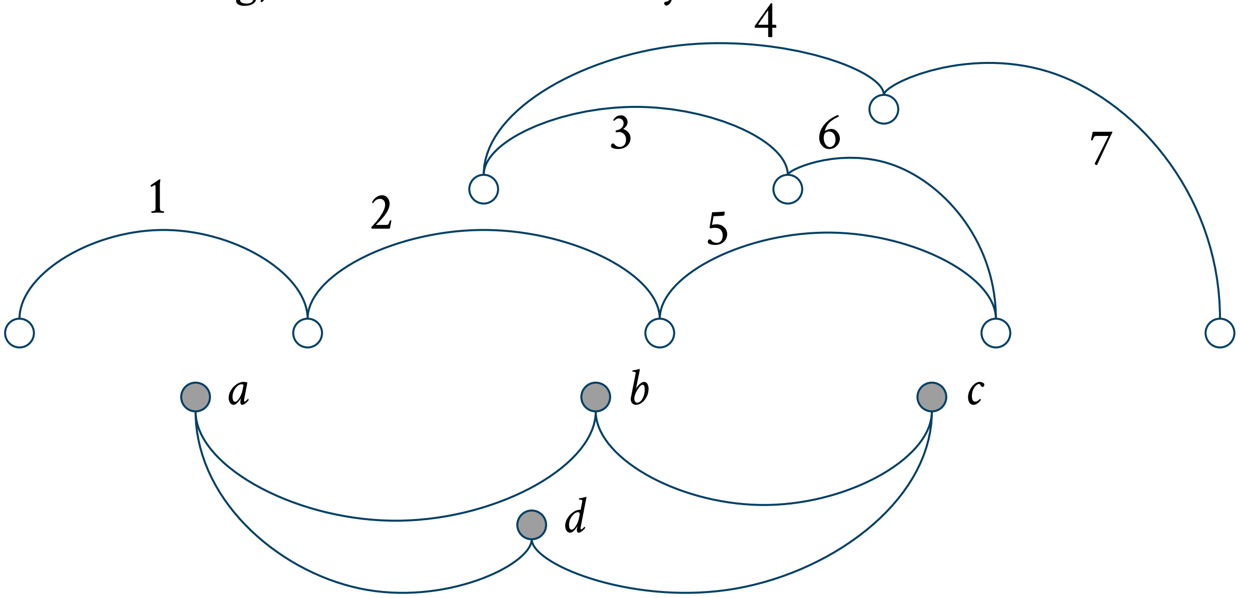
Let G be an *ordered graph* on the vertices $\{1, 2, \dots, n\}$ consisting of two *disjoint components*, a *path* P starting in 1 and ending in n, and a *cycle* C. If G does not contain a *nesting*, then C must be a cycle with *even* number of vertices.



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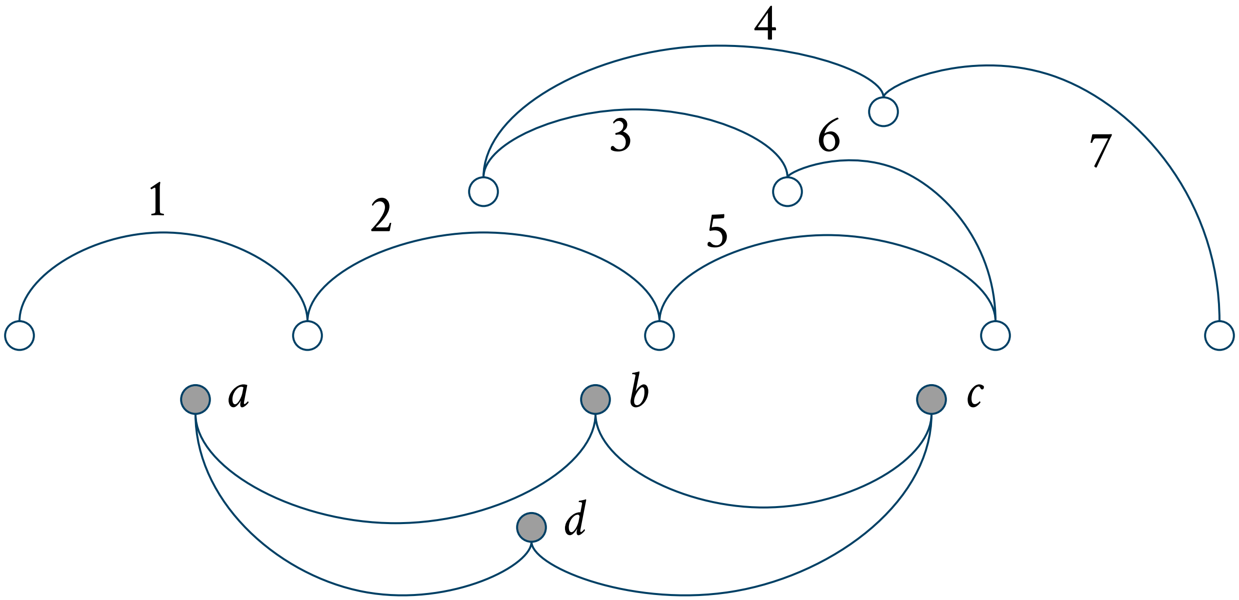

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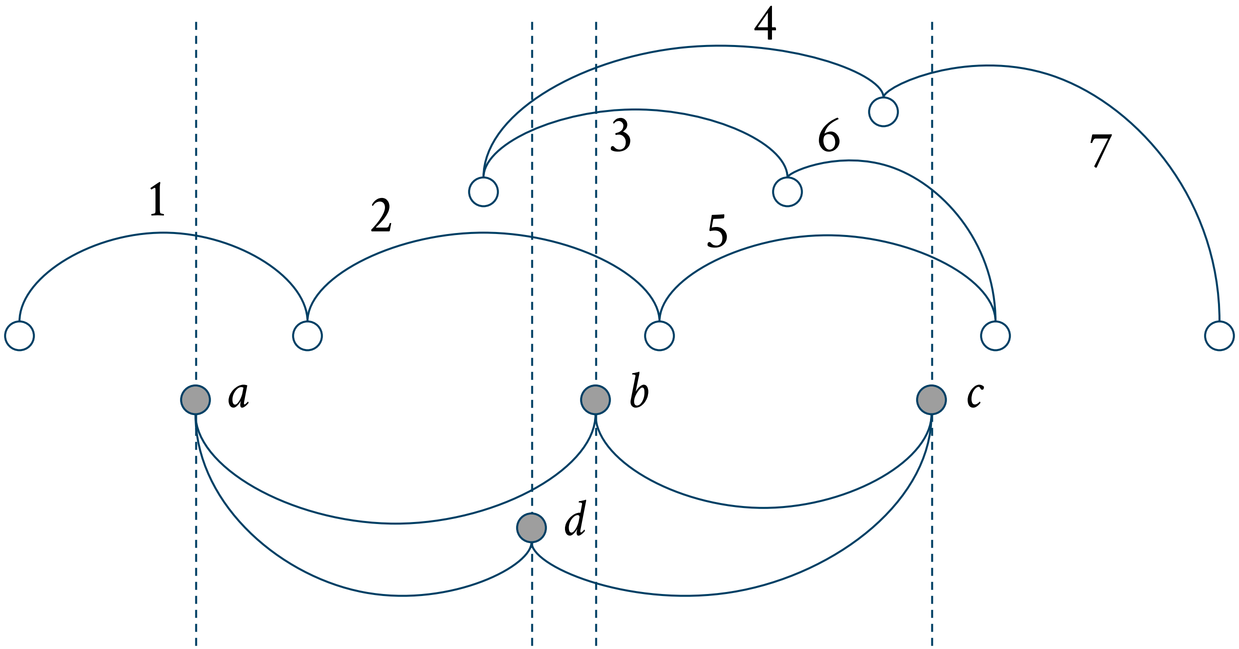

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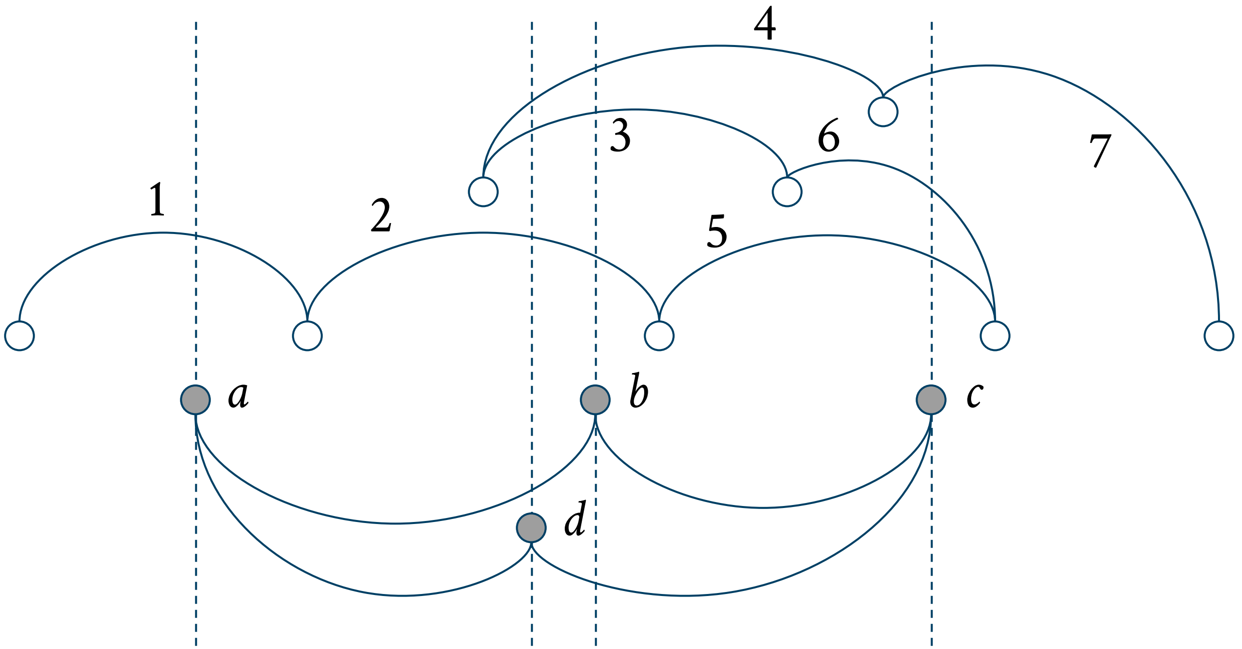

a	*b, d*	*c*
1	2 3 4	5 6 7

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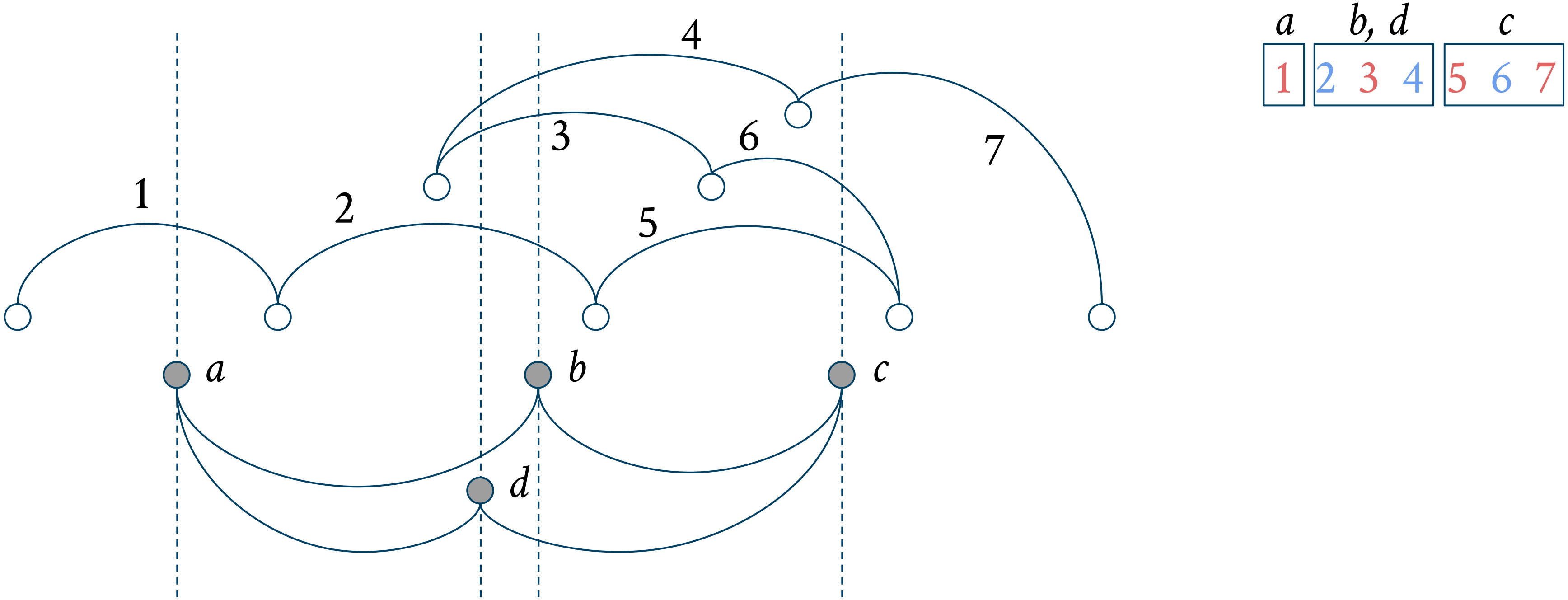

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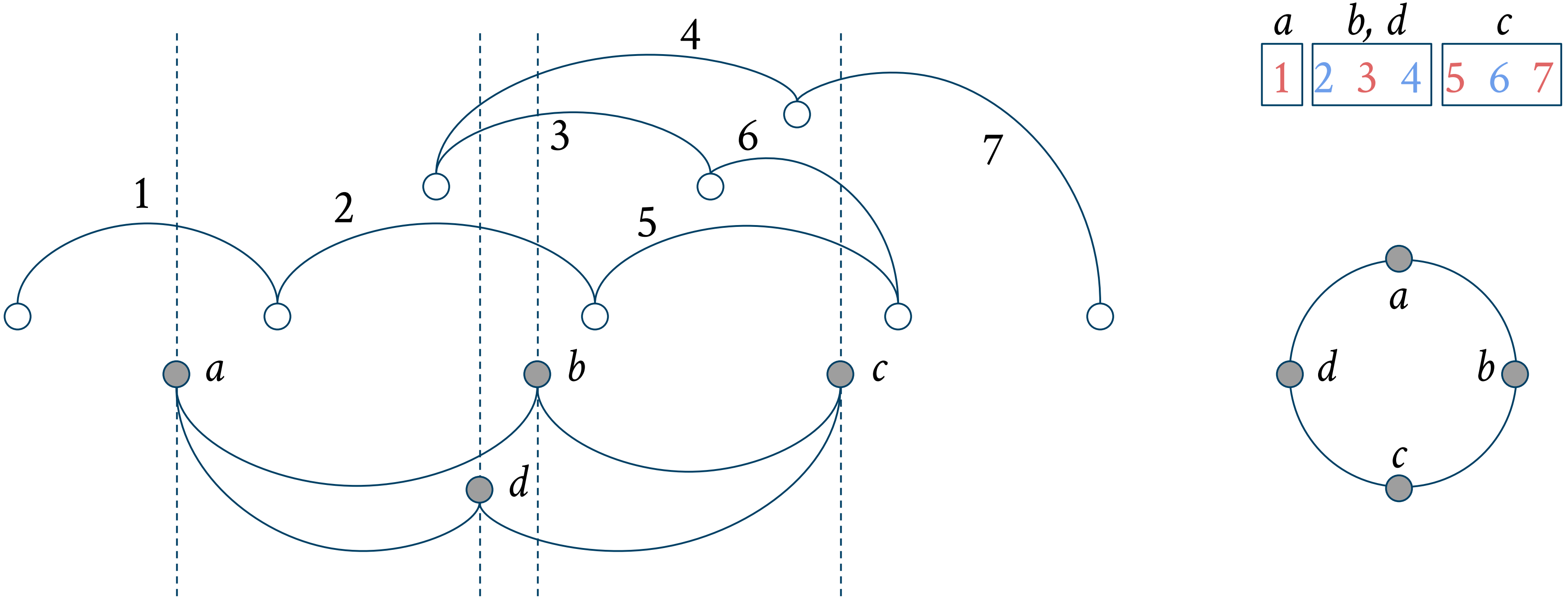

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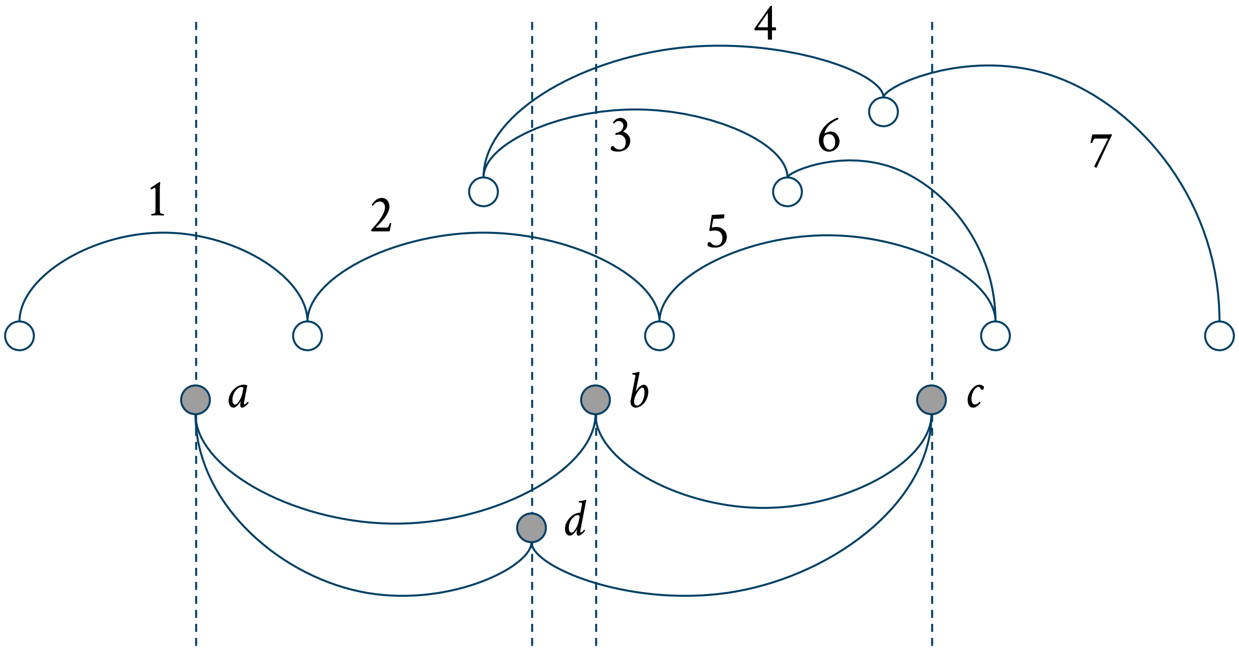

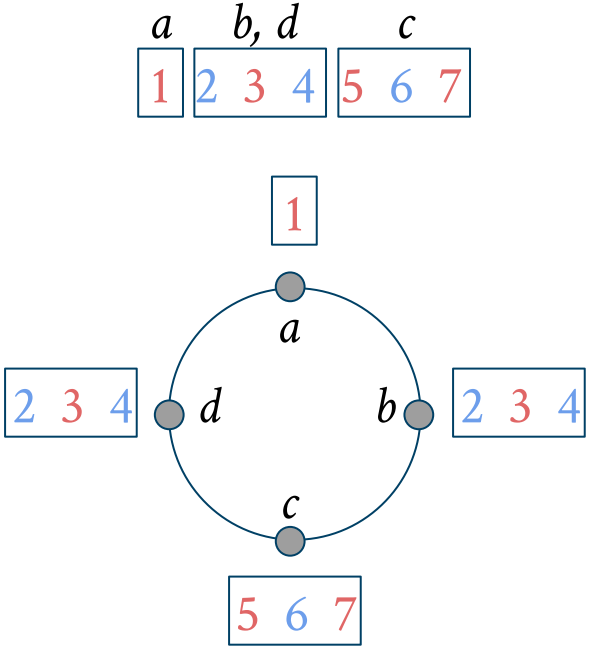

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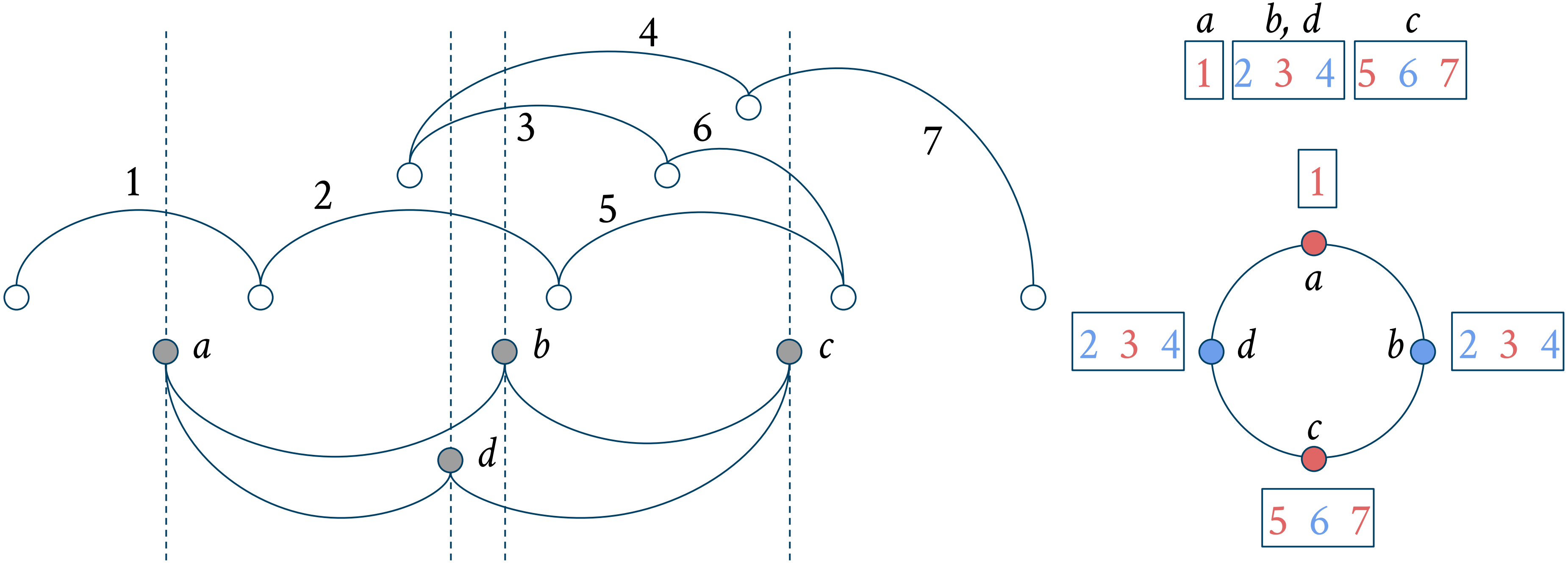

The theorem

Theorem (Grytczuk, Pawlik, Ruciński 2025):

The *odd* power of ABBA is *not* a shuffle square.

A BB AA BB AA BB AA BB AA BB AA BB A

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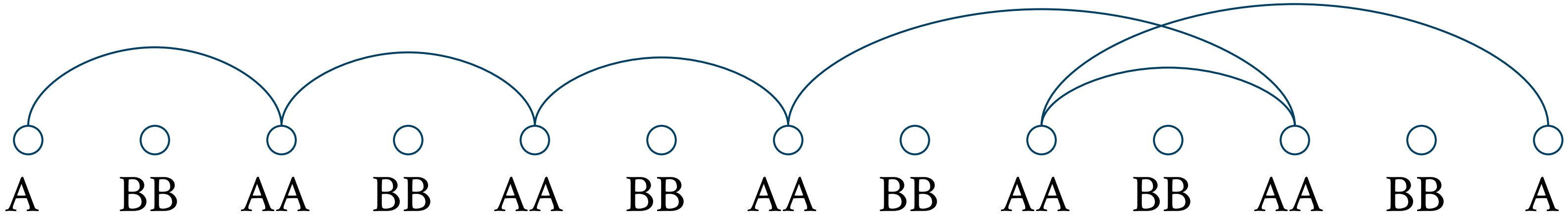

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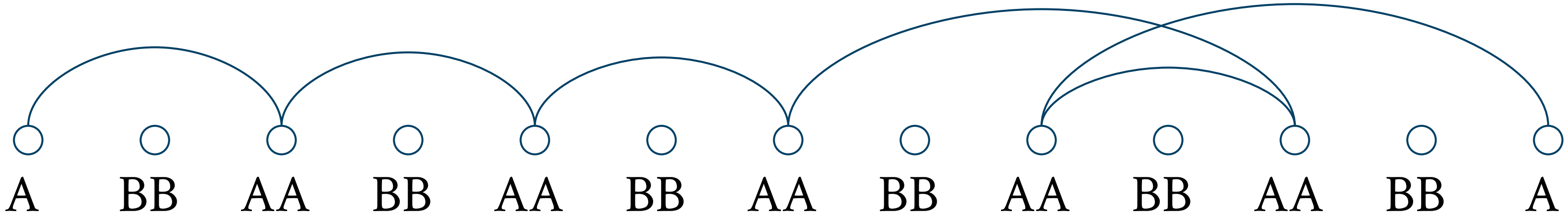

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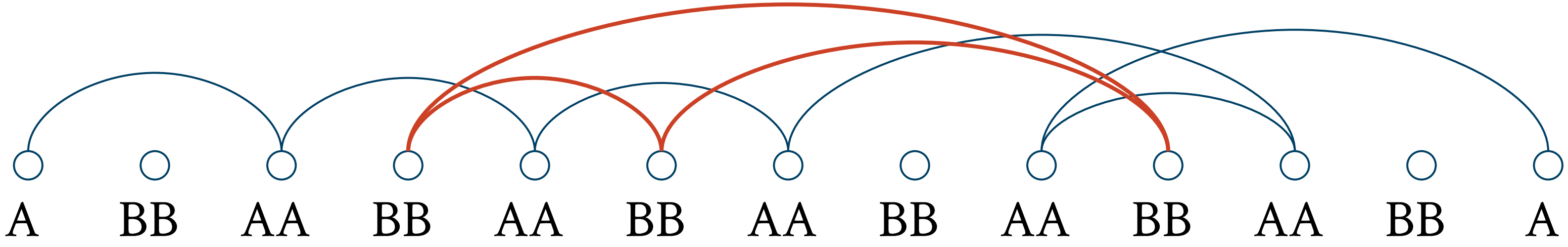

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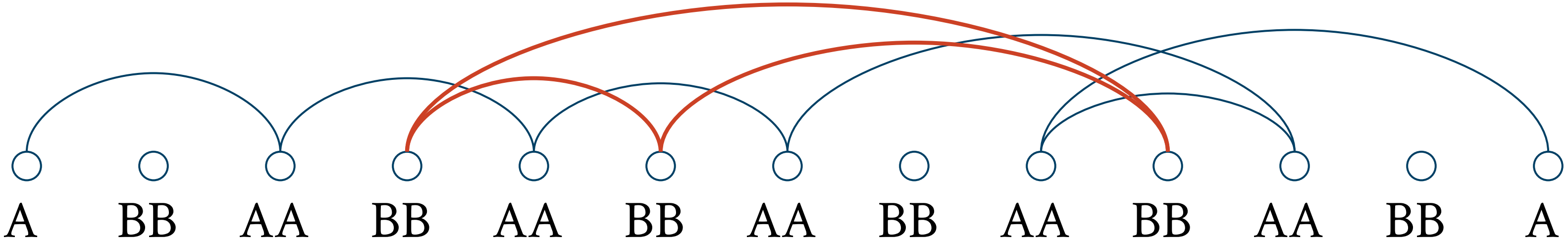

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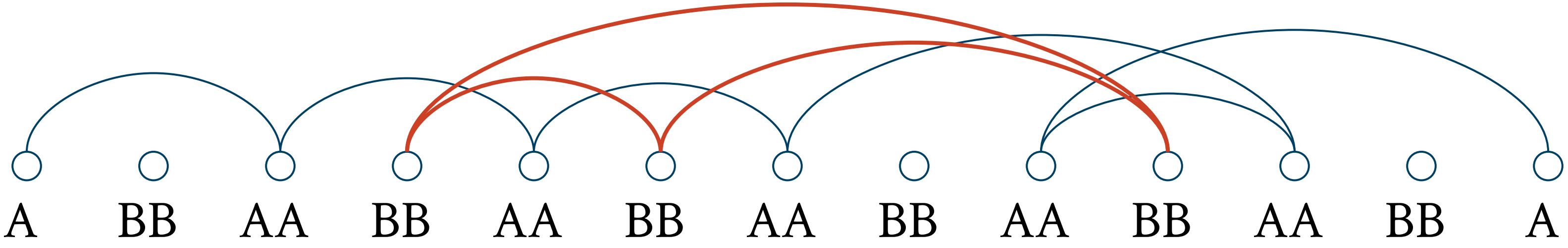

1) There must be a *path* P joining the first A to the last A.
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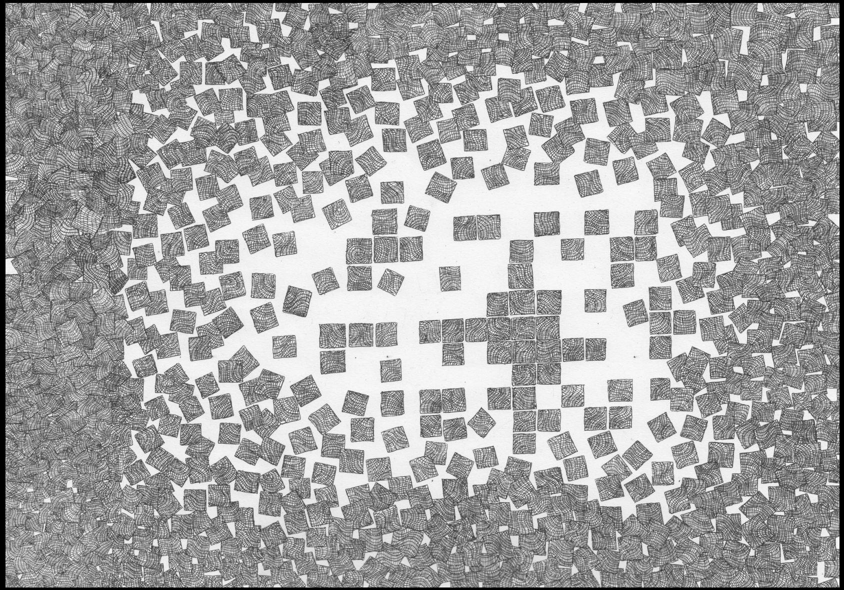
Theorem (Grytczuk, Pawlik, Ruciński 2025):

Let $X = (a(1), a(2), \ldots , a(n{+}1))$ and $Y = (b(1), b(2), \ldots , b(n))$ be two sequences of positive integers such that:

1) All terms of X are *even*, except $a(1)$ and $a(n{+}1)$ which are *odd*.
2) The sequence Y *cannot* be split into *two nonempty* parts with the same *sum*.

Then the word $\mathrm{A}^{a(1)}\,\mathrm{B}^{b(1)}\,\mathrm{A}^{a(2)}\,\mathrm{B}^{b(2)} \ldots \mathrm{A}^{a(n)}\,\mathrm{B}^{b(n)}\,\mathrm{A}^{a(n+1)}$ is *not* a shuffle square.



Thank you!