Positionality of Dumont-Thomas numeration systems

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Numeration systems

Definition

Numeration system over the *domain* $\mathbb{D} \in \{\mathbb{N}, \mathbb{Z}\}$:

- *representation map* $\mathsf{rep}\colon \mathbb{D} \to A^*$
- *evaluation map* $\mathsf{val}\colon L \to \mathbb{D}$, where $\mathsf{rep}(\mathbb{D}) \subseteq L \subseteq A^*$

such that $\mathsf{val} \circ \mathsf{rep} = id_{\mathbb{D}}$.

Common ways of defining numeration systems:

- *Positional* numeration systems: given $(U_n)_{n\in\mathbb{N}}$, set $\mathsf{val}_U\colon A^* \to \mathbb{N} : w_{\ell-1} \cdots w_0 \mapsto \sum_{i=0}^{\ell-1} w_i U_i$.
- *Abstract* numeration systems: if A is ordered, given a regular language $L \subseteq A^*$, val_L is the unique increasing bijection between $(L, \preccurlyeq_{\text{rad}})$ and $(\mathbb{N}, \leq)$.

Positionality (1)

Abstract numeration systems may or may not be also definable in a positional way.

Example

The abstract numeration system defined from $L = \{\mathtt{0}, \dots, \mathtt{9}\}^* \setminus \mathtt{0}\{\mathtt{0}, \dots, \mathtt{9}\}^*$ is equal to the usual decimal system, corresponding to $U_i = 10^i$.

Example

The abstract numeration system defined from $L = \mathtt{1}^*\mathtt{2}^*$ verifies $\mathrm{rep}(3) = \mathtt{11}$ and $\mathrm{rep}(5) = \mathtt{22}$, so it cannot be defined as a positional numeration system.

Positionality (2)

Definition

A numeration system over $\mathbb{N}$ is *positional* if the underlying alphabet A is a set of consecutive integers $\{0, 1, \ldots, c\}$ for some $c \in \mathbb{N}$ and if there exists a sequence $(U_i)_{i\geq 0} \in \mathbb{N}^{\mathbb{N}}$ such that the evaluation map is of the form
$\mathrm{val}\colon A^* \to \mathbb{N}, w_{k-1} \cdots w_0 \mapsto \sum_{i=0}^{k-1} w_i U_i$.

Positionlaity (2)

Definition

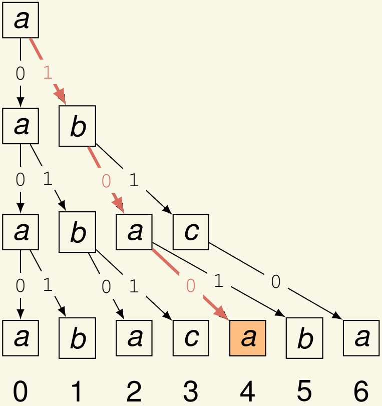
A numeration system over $\mathbb{Z}$ is *positional* if the underlying alphabet A is a set of consecutive integers $\{\mathtt{0}, \mathtt{1}, \ldots, \mathtt{c}\}$ for some $\mathtt{c} \in \mathbb{N}$ and if there exist sequences $(U_i)_{i\geq 0}, (V_i)_{i\geq 0} \in \mathbb{N}^{\mathbb{N}}$ such that the evaluation map is of the form
$\mathrm{val}\colon A^* \to \mathbb{Z}, w_{k-1} \cdots w_0 \mapsto \sum_{i=0}^{k-2} w_i U_i - w_{k-1} V_{k-1}$.

Example

The two's complement numeration system is defined by the weights $U_n = V_n = 2^n$. For instance, we have
$\mathrm{val}(\mathtt{1011}) = -8 + 2 + 1 = -5$.

Dumont-Thomas numeration systems (1989)

$$\mu : \begin{cases} a & \mapsto & ab \\ b & \mapsto & ac \\ c & \mapsto & a \end{cases}, \text{ fixed point starting with } a.$$

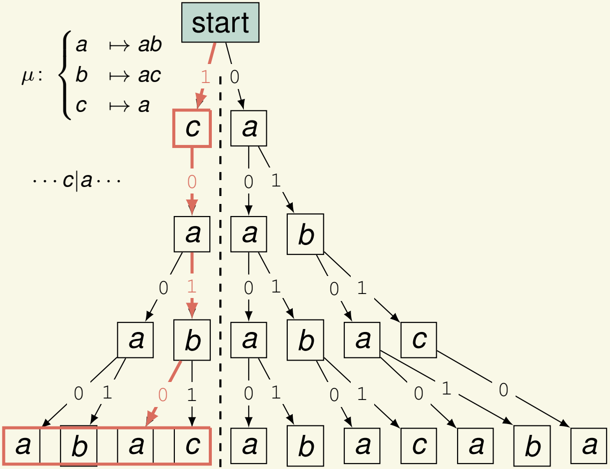


Use the label of a shortest path from the root to column n to represent n: $\mathrm{rep}(4) = \texttt{100}$.

Motivation: use Dumont-Thomas numeration systems as a way to generate Wang tilings $\rightsquigarrow$ we must be able to deal with negative numbers$\rightsquigarrow$ we must be able to deal with periodic points of substitutions.

$$\mu : \begin{cases} a & \mapsto ab \\ b & \mapsto ac \\ c & \mapsto a \end{cases}$$

$\cdots c|a \cdots$



n	rep(n)
7	0001000
6	0110
5	0101
4	0100
3	0011
2	0010
1	0001
0	0
-1	1
-2	1010
-3	1001
-4	1000
-5	1010110

Generalized Dumont-Thomas numerations (2)

Definition (Labbé, Lepšová, 2024; K., Labbé, Stipulanti, 2025)

Given a substitution μ, a periodic point u of period p of μ with seed $b|a$ and some $r \in \{0, \ldots, p-1\}$, the *complement Dumont-Thomas numeration system associated with μ, u and r* is defined by its representation map, such that $\mathrm{rep}(n)$ is the label of a shortest path of length congruent to $r \bmod p$, going from the root to a node in column n in the tree $\mathcal{T}_{\mu,b|a}$.

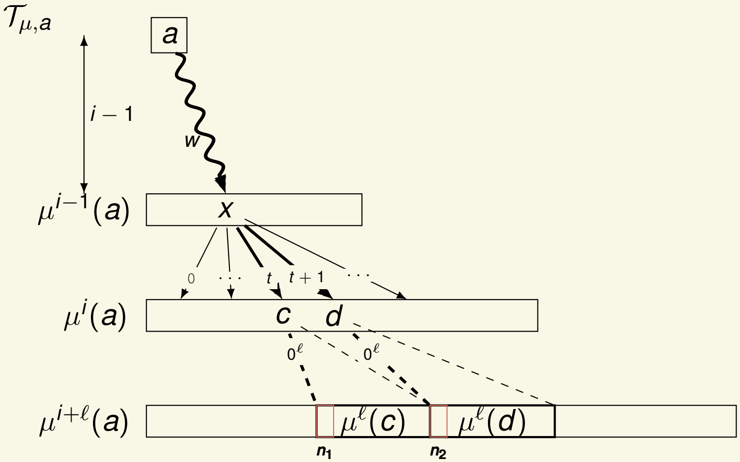
Question

Given μ, u, r, can we decide the positionality of the associated Dumont-Thomas numeration system?

If we consider $\mu\colon a \mapsto aab,\ b \mapsto a$ and $\rho\colon a \mapsto abb,\ b \mapsto ab$, both with seed $b|a$, the system associated with μ is positional but the one associated with ρ is not.

Sketch of the argument (1)

Assume the system is positional and let us compare the values of the words $wt0^\ell$ and $w(t+1)0^\ell$.



$\mathrm{val}(wt0^\ell) = n_1$, $\mathrm{val}(w(t+1)0^\ell) = n_2$, so $U_\ell = n_2 - n_1 = |\mu^\ell(c)|$.

Sketch of the argument (2)

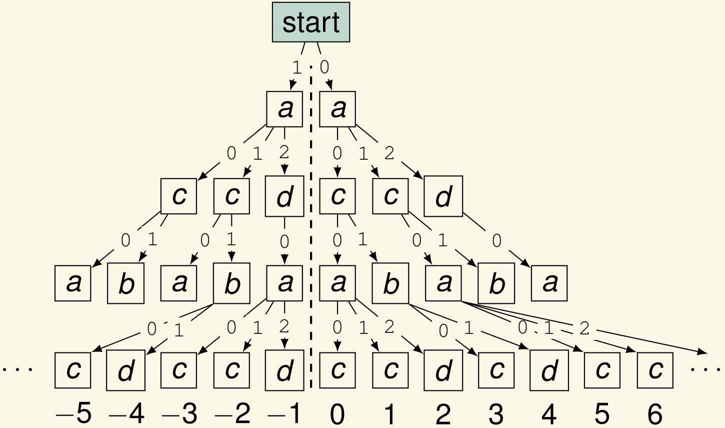
Conjecture (K., Labbé, Stipulanti, 2024)

If the Dumont–Thomas numeration system associated with μ, u, and r is positional, then $\left|\mu^{\ell}(c)\right| = U_{\ell}$ for every letter c that has a younger sibling in $\mathcal{T}_{\mu,b|a}$.

Other direction: If $\left|\mu^{\ell}(c)\right| = U_{\ell}$ for every letter c that has a younger sibling in $\mathcal{T}_{\mu,b|a}$, then incrementing the digit at position ℓ in an expansion increases the value by U_{ℓ}, so the system is indeed positional.

Complications (1)

μ: $a \mapsto ccd, b \mapsto cd, c \mapsto ab, d \mapsto a$, seed $a|a$.

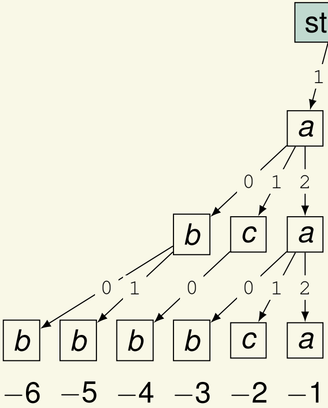


The system is positional for both values of r, despite the fact that $|\mu^\ell(a)| \neq |\mu^\ell(c)|$ for all odd ℓ:

ℓ	0	1	2	3	4
$\lvert\mu^\ell(a)\rvert$	1	3	5	13	21
$\lvert\mu^\ell(c)\rvert$	1	2	5	8	21

Complications (2)

$\mu\colon a \mapsto bca,\ b \mapsto bb,\ c \mapsto b$ with the seed $a|b$.



n	$\mathrm{rep}_{u,0}(n)$
-1	1
-2	11
-3	10
-4	110
-5	101
-6	100
-7	1101
weight	$\cdots 421$

Positional ($U_\ell = 2^\ell,\ V_0 = 1,\ V_\ell = 3.2^\ell$), despite the fact that $\left|\mu^\ell(b)\right| \neq \left|\mu^\ell(c)\right|$ for all $\ell \geq 1$.

Complications (3)

In the first example, the letters a and c occur only at levels of a given parity in the tree, so the sketch can only be applied for half the values of ℓ.

In the second example, the letter to the right of a c can never be part of a shortest path to a column, so the sketch cannot be applied to c.

Definition (Technical)

For $j \in \{0, \ldots, p-1\}$, the set E_j is the set of letters c verifying one of the following conditions:

- There exists a node in $\mathcal{T}_{\mu,b|a}$ labeled by c at a level congruent to $j \bmod p$, in a column other than -2, that has a younger sibling.
- The node at level j and column -2 in $\mathcal{T}_{\mu,b|a}$ is labeled by c, it has a younger sibling labeled by d, and $\left|\mu^{p-j}(d)\right| > 1$.

Main result

We let μ be a substitution, u a periodic point with period p and seed $b|a$, and $r \in \{0, \ldots, p-1\}$.

Theorem (K., Labbé, Stipulanti 2025)

The Dumont-Thomas numeration system associated with μ, u and r is positional if and only if both of the following occur:

- The map $c \mapsto \left|\mu^{\ell}(c)\right|$ has a constant value U_{ℓ} over E_j for all ℓ, j such that $\ell + j \equiv r \bmod p$.
- (Technical) For $j \in \{0, \ldots, r-1\}$, if the node at level j and column -2 in $\mathcal{T}_{\mu,b|a}$ is labeled by c and has a younger sibling labeled by d with $\left|\mu^{p-j}(d)\right| = 1$, then $\left|\mu^{r-j}(c)\right| = U_{r-j}$.

In this case, $(U_{\ell})_{\ell}$ and $V_{\ell} = \left|\mu^{\ell}(b)\right|$ are the sequences of weights of the system.

Corollaries (1)

In some cases, the complications outlined above do not occur.
The first is the case of a primitive substitution.
We say that a letter is *non-final* in the substitution μ if it occurs in the image of any letter at any position other than the last one.

Corollary

Let μ be a primitive substitution, u be a periodic point of μ with seed $b|a$ and period p, and $r \in \{0, \ldots, p-1\}$. The Dumont-Thomas complement system associated with μ, u and r is positional if and only if the map $c \mapsto \left|\mu^\ell(c)\right|$ is constant over the non-final letters in μ for every ℓ.
In this case, the value of the constant is U_ℓ, and $V_\ell = \left|\mu^\ell(b)\right|$.

Corollaries (2)

The second case where no complication occurs is that of the original Dumont-Thomas numeration systems.

Corollary

Let μ be a substitution and u be a right-infinite fixed point of μ. The Dumont-Thomas numeration system associated with μ and u is positional if and only if the map $c \mapsto \left|\mu^{\ell}(c)\right|$ is constant over the non-final letters in μ for every ℓ.
In this case, the value of the constant is U_{ℓ}.

Example

Consider the substitution $\mu\colon a \mapsto abc$, $b \mapsto aac$, $c \mapsto a$. The non-final letters are a and b, and we can show by induction that their images by μ^{ℓ} have the same length for any ℓ. Thus, the system is positional.

Bertrand numeration systems

Bertrand numeration systems (Bertrand-Mathis, 1989; Charlier, Cisternino, Stipulanti, 2022) are a special case of greedy positional numeration systems, defined by either of the two properties:

- $w \in \mathrm{rep}(\mathbb{N}) \Leftrightarrow w0 \in \mathrm{rep}(\mathbb{N})$ for any nonempty w.
- The lexicographically largest words of each length in $\mathrm{rep}(\mathbb{N})$ are all prefixes of one another.

There are three kinds of Bertrand numeration systems:

- $U_\ell = \ell + 1$ (trivial).
- $U_\ell = d_1 U_{\ell-1} + d_2 U_{\ell-2} + \ldots + d_\ell U_0 + 1$ where $d_1 d_2 \cdots$ is the quasi-greedy Rényi representation of 1 in some base β (canonical).
- $U_\ell = d_1 U_{\ell-1} + d_2 U_{\ell-2} + \ldots + d_\ell U_0 + 1$ where $d_1 d_2 \cdots$ is the greedy Rényi representation of 1 in some simple Parry base β (non-canonical).

The original Dumont-Thomas numeration systems have the above properties.

Simplifying the morphism

Let us study the case where μ has a right-infinite fixed point u.

Lemma

Let $\mu\colon A^* \to A^*$ be a substitution such that the map $c \mapsto \left|\mu^\ell(c)\right|$ is constant over the non-final letters in μ for every integer $\ell \geq 0$. Then there exist an alphabet $B \subseteq A$, a substitution $\nu\colon B^* \to B^*$ such that ν has only one non-final letter and a coding $\phi\colon A \to B$ such that ν and $\phi(u)$ define the same Dumont-Thomas numeration system as μ and u.

A substitution that has a fixed point and only one non-final letter is of the form

$$\mu : a_1 \mapsto a_1^{d_1} a_2, a_2 \mapsto a_1^{d_2} a_3, \ldots, a_n \mapsto a_1^{d_n} a_k$$

for some $n \geq 1$, $1 \leq k \leq n$, $d_1 > 0$ and $d_2, \ldots, d_n \geq 0$.

Fabre substitutions

Another approach to Bertrand numeration systems (Fabre, 1995): If β is a Parry number and the quasi-greedy Rényi representation of 1 is $(d_1 \cdots d_{k-1})(d_k \cdots d_n)^\omega$, define the substitution

$$\mu_\beta\colon 1 \mapsto 1^{d_1}2,\ 2 \mapsto 1^{d_2}3 \ldots, n \mapsto 1^{d_n}k.$$

For instance, for β equal to the positive root of $x^3 - x^2 - x - 1 = 0$, the quasi-greedy representation of 1 is $(110)^\omega$ and we find

$$\mu_\beta\colon 1 \mapsto 12,\ 2 \mapsto 13, 3 \mapsto 1.$$

Theorem (Fabre, 1995)

$\left|\mu_\beta^\ell(1)\right|$ is the U_ℓ defined for the canonical Bertrand numeration system for β. This numeration system is the Dumont-Thomas numeration system associated with μ_β and $\mu_\beta^\infty(1)$.

Equivalence between the two systems (1)

The other two kinds of Bertrand numeration systems are also associated with Dumont-Thomas numeration systems when β is Parry.
The converse is not always true.

Example

Consider $\mu\colon$ *a* $\mapsto$ *aab*, *b* $\mapsto$ *aaaa*. The associated Dumont-Thomas system is positional, with weight sequence $1, 3, 10, 32, \ldots$. However, it is not equal to a Bertrand numeration system: $\mathrm{rep}(9) = \texttt{23}$, but it should be $\texttt{30}$.
If such a system were Bertrand associated with some β, the quasi-greedy β-representation of 1 would be $(23)^\omega$, which breaks the Parry conditions.

Equivalence between the two systems (2)

Proposition (K., Labbé, Stipulanti 2025)

Let μ be a substitution of the form

$$\mu : a_1 \mapsto a_1^{d_1} a_2, a_2 \mapsto a_1^{d_2} a_3, \ldots, a_n \mapsto a_1^{d_n} a_k$$

for some $n \geq 1$, $1 \leq k \leq n$, $d_1 > 0$ and $d_2, \ldots, d_n \geq 0$. Construct the word $d_1 d_2 \cdots = d_1 \cdots d_{k-1}(d_k \cdots d_n)^\omega$. The Dumont–Thomas numeration system associated with μ and the seed a_1 is equal to a Bertrand numeration system if and only if we have $d_i d_{i+1} \cdots \preccurlyeq_{lex} d_1 d_2 \cdots$ for each $i \geq 1$.

Conclusion

- Dumont-Thomas numeration systems, interpreting substitutions as trees, are easily generalized to biinfinite periodic points of substitutions.
- Their positionality is completely understood.
- In the case of the original Dumont-Thomas numeration systems, the positional systems are those that correspond to substitutions "in the style of Fabre".
- Those systems are Bertrand numerations associated with Parry numbers if and only if a lexicographic condition is met.

Thank you!