Avoiding abelian and additive powers in rich words

Jonathan Andrade

University of Victoria

Lucas Mol

Thompson Rivers University

WORDS 2025 Conference
Loria, Nancy, France
July 3, 2025

Three types of powers

Three types of powers

- An (ordinary) square is a nonempty word of the form xx.
  - E.g., `murmur`, `hotshots`

Three types of powers

- An (ordinary) square is a nonempty word of the form xx.
  - E.g., `murmur`, `hotshots`
- An abelian square is a nonempty word of the form xx', where x' is an *anagram* of x.
  - E.g., `reappear`, `intestines`

Three types of powers

- An (ordinary) square is a nonempty word of the form xx.
  - E.g., `murmur`, `hotshots`
- An abelian square is a nonempty word of the form xx', where x' is an *anagram* of x.
  - E.g., `reappear`, `intestines`
- An additive square is a nonempty word of the form xx', where x and x' have the same length and the same sum.
  - E.g., `0312`, `210201`

Three types of powers

- An (ordinary) square is a nonempty word of the form xx.
  - E.g., `murmur`, `hotshots`
- An abelian square is a nonempty word of the form xx', where x' is an *anagram* of x.
  - E.g., `reappear`, `intestines`
- An additive square is a nonempty word of the form xx', where x and x' have the same length and the same sum.
  - E.g., 0312, 210201
- Ordinary, abelian, and additive k-powers are defined analogously for $k \geq 2$.
  - E.g., 011 101 110 011 is an additive 4-power.

Avoiding abelian powers

Avoiding abelian powers

Theorem (Dekking 1979): Abelian 4-powers are avoidable over $\{0, 1\}$.

Avoiding abelian powers

Theorem (Dekking 1979): Abelian 4-powers are avoidable over $\{0, 1\}$.

Theorem (Dekking 1979): Abelian cubes are avoidable over $\{0, 1, 2\}$.

Avoiding abelian powers

Theorem (Dekking 1979): Abelian 4-powers are avoidable over $\{0, 1\}$.

Theorem (Dekking 1979): Abelian cubes are avoidable over $\{0, 1, 2\}$.

Theorem (Keränen 1992): Abelian squares are avoidable over $\{0, 1, 2, 3\}$.

Avoiding abelian powers

Theorem (Dekking 1979): Abelian 4-powers are avoidable over $\{0, 1\}$.

Theorem (Dekking 1979): Abelian cubes are avoidable over $\{0, 1, 2\}$.

Theorem (Keränen 1992): Abelian squares are avoidable over $\{0, 1, 2, 3\}$.

- All three alphabet sizes are smallest possible.

Avoiding additive powers

Avoiding additive powers

Theorem (Dekking 1979): Additive 4-powers are avoidable over $\{0, 1\}$.

Avoiding additive powers

Theorem (Dekking 1979): Additive 4-powers are avoidable over $\{0, 1\}$.

Theorem (Cassaigne, Currie, Schaeffer, and Shallit 2014): Additive cubes are avoidable over $\{0, 1, 3, 4\}$.

Avoiding additive powers

Theorem (Dekking 1979): Additive 4-powers are avoidable over $\{0, 1\}$.

Theorem (Cassaigne, Currie, Schaeffer, and Shallit 2014): Additive cubes are avoidable over $\{0, 1, 3, 4\}$.

Theorem (Rao 2015): Additive cubes are avoidable over several subsets of $\mathbb{Z}$ of size 3.

Avoiding additive powers

Theorem (Dekking 1979): Additive 4-powers are avoidable over $\{0, 1\}$.

Theorem (Cassaigne, Currie, Schaeffer, and Shallit 2014): Additive cubes are avoidable over $\{0, 1, 3, 4\}$.

Theorem (Rao 2015): Additive cubes are avoidable over several subsets of $\mathbb{Z}$ of size 3.

Theorem (Lietard and Rosenfeld 2020): Additive cubes are avoidable over all subsets of $\mathbb{Z}$ of size 4, except possibly those equivalent to $\{0, 1, 2, 3\}$.

Avoiding additive powers

Theorem (Dekking 1979): Additive 4-powers are avoidable over $\{0, 1\}$.

Theorem (Cassaigne, Currie, Schaeffer, and Shallit 2014): Additive cubes are avoidable over $\{0, 1, 3, 4\}$.

Theorem (Rao 2015): Additive cubes are avoidable over several subsets of $\mathbb{Z}$ of size 3.

Theorem (Lietard and Rosenfeld 2020): Additive cubes are avoidable over all subsets of $\mathbb{Z}$ of size 4, except possibly those equivalent to $\{0, 1, 2, 3\}$.

Open Problem: Are additive squares avoidable over some finite subset of $\mathbb{Z}$?

Rich words

Rich words

- Question: How many distinct nonempty palindromes can a word of length n contain?

Rich words

- Question: How many distinct nonempty palindromes can a word of length n contain?
- For insight, let's build the word `0010120` one letter at a time.

Rich words

- Question: How many distinct nonempty palindromes can a word of length n contain?
- For insight, let's build the word `0010120` one letter at a time.

`0`

Rich words

- Question: How many distinct nonempty palindromes can a word of length n contain?
- For insight, let's build the word `0010120` one letter at a time.

`0`

Rich words

- Question: How many distinct nonempty palindromes can a word of length n contain?
- For insight, let's build the word `0010120` one letter at a time.

`0`
`00`

Rich words

- Question: How many distinct nonempty palindromes can a word of length n contain?
- For insight, let's build the word `0010120` one letter at a time.

`0`
`00`

Rich words

- Question: How many distinct nonempty palindromes can a word of length n contain?
- For insight, let's build the word `0010120` one letter at a time.

`0`
`00`
`001`

Rich words

- Question: How many distinct nonempty palindromes can a word of length n contain?
- For insight, let's build the word 0010120 one letter at a time.

0
00
001

Rich words

- Question: How many distinct nonempty palindromes can a word of length n contain?
- For insight, let's build the word 0010120 one letter at a time.

0
00
001
0010

Rich words

- Question: How many distinct nonempty palindromes can a word of length n contain?
- For insight, let's build the word 0010120 one letter at a time.

0
00
001
0010

Rich words

- Question: How many distinct nonempty palindromes can a word of length n contain?
- For insight, let's build the word `0010120` one letter at a time.

```
0
00
001
0010
00101
```

Rich words

- Question: How many distinct nonempty palindromes can a word of length n contain?
- For insight, let's build the word 0010120 one letter at a time.

0
00
001
0010
00101

Rich words

- Question: How many distinct nonempty palindromes can a word of length n contain?
- For insight, let's build the word `0010120` one letter at a time.

```
0
00
001
0010
00101
001012
```

Rich words

- Question: How many distinct nonempty palindromes can a word of length n contain?
- For insight, let's build the word 0010120 one letter at a time.

```
0
00
001
0010
00101
001012
```

Rich words

- Question: How many distinct nonempty palindromes can a word of length n contain?
- For insight, let's build the word 0010120 one letter at a time.

```
0
00
001
0010
00101
001012
0010120
```

Rich words

- Question: How many distinct nonempty palindromes can a word of length n contain?
- For insight, let's build the word 0010120 one letter at a time.

0
00
001
0010
00101
001012
0010120

Theorem (Droubay, Justin, and Pirillo 2001): Every word of length n contains at most n distinct nonempty palindromes.

Rich words

- Question: How many distinct nonempty palindromes can a word of length n contain?
- For insight, let's build the word 0010120 one letter at a time.

0
00
001
0010
00101
001012
0010120

Theorem (Droubay, Justin, and Pirillo 2001): Every word of length n contains at most n distinct nonempty palindromes.

- A word of length n is rich if it contains n distinct nonempty palindromes.

Rich words

- Question: How many distinct nonempty palindromes can a word of length n contain?
- For insight, let's build the word 0010120 one letter at a time.

0
00
001
0010
00101
001012
0010120

Theorem (Droubay, Justin, and Pirillo 2001): Every word of length n contains at most n distinct nonempty palindromes.

- A word of length n is rich if it contains n distinct nonempty palindromes.
  - E.g., 001012 is rich, but 0010120 is not.

Rich words

- Question: How many distinct nonempty palindromes can a word of length n contain?
- For insight, let's build the word 0010120 one letter at a time.

0
00
001
0010
00101
001012
0010120

Theorem (Droubay, Justin, and Pirillo 2001): Every word of length n contains at most n distinct nonempty palindromes.

- A word of length n is rich if it contains n distinct nonempty palindromes.
  - E.g., 001012 is rich, but 0010120 is not.
- An infinite word is rich if all of its finite factors are rich.

Avoiding abelian and additive powers in rich words

Avoiding abelian and additive powers in rich words

Theorem (Andrade and Mol 2025): There is an infinite additive 5-power-free rich word over $\{0, 1\}$.

Avoiding abelian and additive powers in rich words

Theorem (Andrade and Mol 2025): There is an infinite additive 5-power-free rich word over $\{0, 1\}$.

Theorem (Andrade and Mol 2025): There is an infinite additive 4-power-free rich word over $\{0, 1, 2\}$.

Avoiding abelian and additive powers in rich words

Theorem (Andrade and Mol 2025): There is an infinite additive 5-power-free rich word over $\{0, 1\}$.

Theorem (Andrade and Mol 2025): There is an infinite additive 4-power-free rich word over $\{0, 1, 2\}$.

- The alphabet sizes are smallest possible, even for abelian powers.

Avoiding abelian and additive powers in rich words

Theorem (Andrade and Mol 2025): There is an infinite additive 5-power-free rich word over $\{0, 1\}$.

Theorem (Andrade and Mol 2025): There is an infinite additive 4-power-free rich word over $\{0, 1, 2\}$.

- The alphabet sizes are smallest possible, even for abelian powers.
- But what about squares and cubes?

Avoiding abelian and additive powers in rich words

Theorem (Andrade and Mol 2025): There is an infinite additive 5-power-free rich word over $\{0, 1\}$.

Theorem (Andrade and Mol 2025): There is an infinite additive 4-power-free rich word over $\{0, 1, 2\}$.

- The alphabet sizes are smallest possible, even for abelian powers.
- But what about squares and cubes?

Theorem (Pelantová and Starosta 2013): Every infinite rich word over a finite alphabet contains an ordinary square.

Avoiding abelian and additive powers in rich words

Theorem (Andrade and Mol 2025): There is an infinite additive 5-power-free rich word over $\{0, 1\}$.

Theorem (Andrade and Mol 2025): There is an infinite additive 4-power-free rich word over $\{0, 1, 2\}$.

- The alphabet sizes are smallest possible, even for abelian powers.
- But what about squares and cubes?

Theorem (Pelantová and Starosta 2013): Every infinite rich word over a finite alphabet contains an ordinary square.

Open Problem: Is there an infinite additive (or abelian) cube-free rich word over some finite subset of $\mathbb{Z}$?

Our constructions

Define $\beta : \{0,1\}^* \to \{0,1\}^*$ and $\gamma : \{0,1,2\}^* \to \{0,1,2\}^*$ by

$$\beta(0) = 00001 \qquad \gamma(0) = 2$$
$$\beta(1) = 01101 \qquad \gamma(1) = 101$$
$$\gamma(2) = 10001$$

Our constructions

Define $\beta : \{0,1\}^* \to \{0,1\}^*$ and $\gamma : \{0,1,2\}^* \to \{0,1,2\}^*$ by

$$\beta(0) = 00001 \qquad \gamma(0) = 2$$
$$\beta(1) = 01101 \qquad \gamma(1) = 101$$
$$\gamma(2) = 10001$$

We show that

- $\mathbf{B} = \beta^\omega(0)$ is rich and additive 5-power-free,

Our constructions

Define $\beta : \{0,1\}^* \to \{0,1\}^*$ and $\gamma : \{0,1,2\}^* \to \{0,1,2\}^*$ by

$$\beta(0) = 00001 \qquad \gamma(0) = 2$$
$$\beta(1) = 01101 \qquad \gamma(1) = 101$$
$$\gamma(2) = 10001$$

We show that

- $\mathbf{B} = \beta^\omega(0)$ is rich and additive 5-power-free, and
- $\mathbf{\Gamma} = \gamma^\omega(1)$ is rich and additive 4-power-free.

Proving richness

We make use of the following characterization.

Theorem (Droubay, Justin, and Pirillo 2001): An infinite word $\mathbf{w}$ is rich if and only if every finite prefix of $\mathbf{w}$ has a unioccurrent palindromic suffix.

Proving richness

We make use of the following characterization.

Theorem (Droubay, Justin, and Pirillo 2001): An infinite word **w** is rich if and only if every finite prefix of **w** has a unioccurrent palindromic suffix.

For **B**, we use `Walnut`.

```
morphism b "0->00001 1->01101":
promote B b:
def FactorEq "?msd_5 Ak (k<n)=>(B[i+k]=B[j+k])":
def Occurs "?msd_5 (m<= n) & ( Ek (k+m<=n) & $FactorEq(i,j+k,m))":
def Palindrome "?msd_5 Aj,k ((k<n) & (j+k+1=n)) => (B[i+k]=B[i+j])":
def BisRich  "?msd_5 An Ej $Palindrome(j,n-j) & ~$Occurs(j,0,n-j,n-1)":
```

Proving richness

We make use of the following characterization.

Theorem (Droubay, Justin, and Pirillo 2001): An infinite word **w** is rich if and only if every finite prefix of **w** has a unioccurrent palindromic suffix.

For **B**, we use `Walnut`.

```
morphism b "0->00001 1->01101":
promote B b:
def FactorEq "?msd_5 Ak (k<n)=>(B[i+k]=B[j+k])":
def Occurs "?msd_5 (m<= n) & ( Ek (k+m<=n) & $FactorEq(i,j+k,m))":
def Palindrome "?msd_5 Aj,k ((k<n) & (j+k+1=n)) => (B[i+k]=B[i+j])":
def BisRich  "?msd_5 An Ej $Palindrome(j,n-j) & ~$Occurs(j,0,n-j,n-1)":
```

For **Γ**, we use an inductive proof with several cases.

Proving additive power-freeness

We use a recent variation of the template method.

Proving additive power-freeness

We use a recent variation of the template method.

Theorem (Currie and Rampersad 2012): There is an algorithm which decides, under certain conditions on h, whether $h^\omega(0)$ contains abelian k-powers.

Proving additive power-freeness

We use a recent variation of the template method.

Theorem (Currie and Rampersad 2012): There is an algorithm which decides, under certain conditions on h, whether $h^\omega(0)$ contains abelian k-powers.

Theorem (Rao and Rosenfeld 2018): Less restrictive conditions, can be used for both abelian and additive powers.

Proving additive power-freeness

We use a recent variation of the template method.

Theorem (Currie and Rampersad 2012): There is an algorithm which decides, under certain conditions on h, whether $h^\omega(0)$ contains abelian k-powers.

Theorem (Rao and Rosenfeld 2018): Less restrictive conditions, can be used for both abelian and additive powers.

Theorem (Currie, Mol, Rampersad, and Shallit 2024): More restrictive conditions, and works only for additive powers, but (probably) simpler and more efficient.

Proving additive power-freeness

We use a recent variation of the template method.

Theorem (Currie and Rampersad 2012): There is an algorithm which decides, under certain conditions on h, whether $h^\omega(0)$ contains abelian k-powers.

Theorem (Rao and Rosenfeld 2018): Less restrictive conditions, can be used for both abelian and additive powers.

Theorem (Currie, Mol, Rampersad, and Shallit 2024): More restrictive conditions, and works only for additive powers, but (probably) simpler and more efficient.

- Jonathan Andrade implemented this algorithm in general, and then applied it to $\mathbf{B} = \beta^\omega(0)$ and $\mathbf{\Gamma} = \gamma^\omega(1)$.

The Essential Idea of the Template Method

The Essential Idea of the Template Method

- Any long abelian/additive power in $h^\omega(0)$ must have arisen by applying h repeatedly to some short "seed word".

The Essential Idea of the Template Method

- Any long abelian/additive power in $h^{\omega}(0)$ must have arisen by applying h repeatedly to some short "seed word".
- These seed words cannot look "too different" from abelian/additive powers – there are only finitely many possible *templates* for these seed words.

The Essential Idea of the Template Method

- Any long abelian/additive power in $h^\omega(0)$ must have arisen by applying h repeatedly to some short "seed word".
- These seed words cannot look "too different" from abelian/additive powers – there are only finitely many possible *templates* for these seed words.
- We can enumerate all short words in $h^\omega(0)$, and check to see if they match any of the templates.

Templates for abelian powers

Templates for abelian powers

- A *template* for abelian squares is a 4-tuple

$$[a_0, a_1, a_2, \vec{d}]$$

letters or ε vector

Templates for abelian powers

- A *template* for abelian squares is a 4-tuple

$$[a_0, a_1, a_2, \vec{d}]$$

letters or ε vector

- A word w is an *instance* of this template if

$$w = a_0 w_0 a_1 w_1 a_2 \quad \text{and} \quad \psi(w_1) - \psi(w_0) = \vec{d}.$$

Templates for abelian powers

- A *template* for abelian squares is a 4-tuple

$$[a_0, a_1, a_2, \vec{d}]$$

letters or ε vector

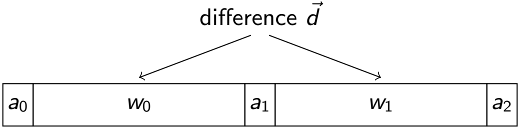
- A word w is an *instance* of this template if

$$w = a_0 w_0 a_1 w_1 a_2 \quad \text{and} \quad \psi(w_1) - \psi(w_0) = \vec{d}.$$

- An instance of $[\varepsilon, \varepsilon, \varepsilon, \vec{0}]$ is an abelian square.

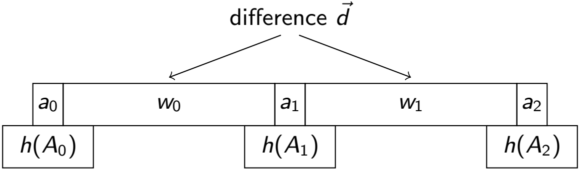
Parents

Every long-enough instance of a template must have come from an instance of another template – a parent.



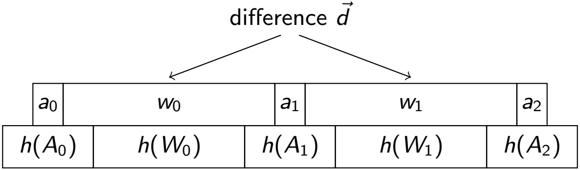
Parents

Every long-enough instance of a template must have come from an instance of another template – a parent.



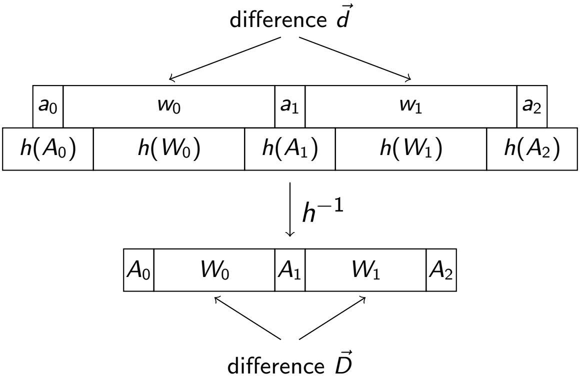
Parents

Every long-enough instance of a template must have come from an instance of another template – a parent.



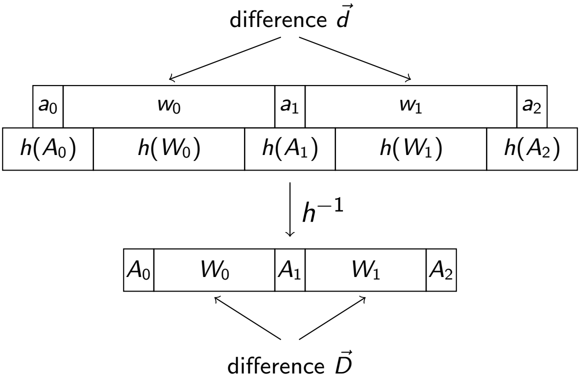
Parents

Every long-enough instance of a template must have come from an instance of another template – a parent.



Parents

Every long-enough instance of a template must have come from an instance of another template – a parent.



Conditions on h facilitate the enumeration of the parents of a given template, and ensure that each template has only finitely many ancestors (parents, grandparents, etc.)

Templates for additive powers

Templates for additive powers

- Idea: For additive powers, we'd like to replace Parikh vectors with vectors having just two entries (length and sum).

Templates for additive powers

- Idea: For additive powers, we'd like to replace Parikh vectors with vectors having just two entries (length and sum).
- Caveat: The morphism must have some special structure.

Templates for additive powers

- Idea: For additive powers, we'd like to replace Parikh vectors with vectors having just two entries (length and sum).
- Caveat: The morphism must have some special structure.
  - $\beta(0) = 00001$ and $\beta(1) = 01101$, so for all $x \in \{0, 1\}$,

$$|\beta(x)| = 5 + 0x \quad \text{and} \quad \sum \beta(x) = 1 + 2x.$$

Templates for additive powers

- Idea: For additive powers, we'd like to replace Parikh vectors with vectors having just two entries (length and sum).
- Caveat: The morphism must have some special structure.
  - $\beta(0) = 00001$ and $\beta(1) = 01101$, so for all $x \in \{0, 1\}$,
    $$|\beta(x)| = 5 + 0x \quad \text{and} \quad \sum \beta(x) = 1 + 2x.$$
  - $\gamma(0) = 2$, $\gamma(1) = 101$, and $\gamma(2) = 10001$, so for all $x \in \{0, 1, 2\}$,
    $$|\gamma(x)| = 1 + 2x \quad \text{and} \quad \sum \gamma(x) = 2 + 0x.$$

Templates for additive powers

- Idea: For additive powers, we'd like to replace Parikh vectors with vectors having just two entries (length and sum).
- Caveat: The morphism must have some special structure.
  - $\beta(0) = 00001$ and $\beta(1) = 01101$, so for all $x \in \{0, 1\}$,
  $$|\beta(x)| = 5 + 0x \quad \text{and} \quad \sum \beta(x) = 1 + 2x.$$
  - $\gamma(0) = 2$, $\gamma(1) = 101$, and $\gamma(2) = 10001$, so for all $x \in \{0, 1, 2\}$,
  $$|\gamma(x)| = 1 + 2x \quad \text{and} \quad \sum \gamma(x) = 2 + 0x.$$
- We say that these morphisms are affine.

Outlook

Outlook

Open Problem: Is there an infinite additive square-free word over some finite subset of $\mathbb{Z}$?

Outlook

Open Problem: Is there an infinite additive square-free word over some finite subset of $\mathbb{Z}$?

Open Problem: Is there an infinite additive (or abelian) cube-free rich word over some finite subset of $\mathbb{Z}$?

Outlook

Open Problem: Is there an infinite additive square-free word over some finite subset of $\mathbb{Z}$?

Open Problem: Is there an infinite additive (or abelian) cube-free rich word over some finite subset of $\mathbb{Z}$?

Open Problem: Is there an infinite additive 4-power-free rich word over *every* subset of $\mathbb{Z}$ of size 3?

Thank you!