And now there are four: Another brick in the wall of the optimal upper bound on the MP-ratio

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WORDS 2025
Nancy, France
June 30, 2025

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- It is a rational number greater than or equal to 1, such that, the larger it is, the more the given word is “palindromic”.
- The concept was originally defined only for binary words, and its extension to larger alphabets was left as an open problem.

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Every binary word is characterized, up to reversal, by the set of its subpalindromes.

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Proof (sketch).

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- $r, s \in \{0, 1, \ldots, n-1\}^*$, (r, s) is an MP-extension of w $\leftrightarrow$ rws is minimal-palindromic; – Does it always exist???
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Let $M = 2^{\lceil \frac{n}{2} \rceil} + 2^{\lfloor \frac{n}{2} \rfloor - 1} - 3$; exceptionally, if $n = 4$ or $n = 5$, we define $M = 4$, respectively $M = 8$ instead.

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where $l_0 = M|w| - |w|_0$ and

$$l_i = \begin{cases} (M - 2^i + 1)|w|, & \text{for } i = 1, \dots, \lceil \frac{n}{2} \rceil - 1; \\ (2^{n-1-i} - 1)|w| - |w|_i, & \text{for } i = \lceil \frac{n}{2} \rceil, \dots, n-2, \end{cases}$$

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The optimal upper bound on the MP-ratio is not less than $2n$.

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Now we know that the optimal upper bound on the MP-ratio for $n \geqslant 4$ is somewhere between

$$2n \quad \text{and} \quad \sim 2^{\frac{n}{2}} n,$$

and it remains an open problem to narrow (or even better, close) this gap.

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Proof (sketch).

$$f(w) = 0^{2|w|-|w|_0} 2^{2|w|-|w|_2-g'(w,0,2)} \; w \; 2^{g'(w,0,2)} 1^{2|w|-|w|_1}$$

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Let $u \in \{1, 2\}^*$, $t, v \in 2^*$ *and let* p *and* q *be subpalindromes of* tu *and* uv*, respectively. If* $|p| + |q| > 2|u|$,

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$$|u|_1 \leqslant \frac{|tv|-1}{|tv|}|tuv|_2.$$

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We have $|f_1(w)| = |f_2(w)| = 8|w|$, and we prove that at least one of the words $f_1(w)$ and $f_2(w)$ does not have a subpalindrome whose length exceeds $2|w|$.

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This will follow by Propositions 1–5. ∎

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Proposition

The length of an arbitrary subpalindrome of the form $0p0$ in each of the words $f_1(w)$ and $f_2(w)$ is less than or equal to $2|w|$.

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Case analysis for $f_1(w)$, and for $f_2(w)$ we use that $f_2(w) = \widetilde{f_1(\widetilde{w})}$. ■

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The length of an arbitrary subpalindrome of the form $0p0$ *in each of the words* $f_1(w)$ *and* $f_2(w)$ *is less than or equal to* $2|w|$.

Proof (sketch).

Case analysis for $f_1(w)$, and for $f_2(w)$ we use that $f_2(w) = \widetilde{f_1(\widetilde{w})}$. ∎

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Propositions 1–2

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Proof.

Analogous to the previous one (where 0s and 1s have switched roles, as well as 2s and 3s, and as well as $f_1(w)$ and $f_2(w)$). ■

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Definition

Let p and q be subpalindromes of some word z, and let P and Q be their descriptors. We say that Q is offset to the right of P if and only if for each s, s' such that $s \in P$, $s' \in Q$, $s' \leqslant s$, we have $\sigma_Q(s') > \sigma_P(s)$.

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$$0^{2|w|-|w|_0}3^{2|w|-|w|_3-g(w,0,3)}2^{g(\widetilde{w},1,2)}\ w\ 3^{g(w,0,3)}2^{2|w|-|w|_2-g(\widetilde{w},1,2)}1^{2|w|-|w|_1};$$

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- $P_{|p_w|} < Q_1$: ✓
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- Q is offset to the right of P: Time to roll up our sleeves.

Setting the stage

$$3^{g(\widetilde{w},0,3)}w3^{g(w,0,3)}$$

Let $w \in \{0,1,2,3\}^*$, and let m_1, m_2 be nonnegative integers such that $m_1 \leqslant 3^{g(\widetilde{w},0,3)}$ and $m_2 \leqslant 3^{g(w,0,3)}$. Let p and q be subpalindromes of $3^{m_1}w$ and $w3^{m_2}$, respectively. Let P and Q be descriptors of p and q (with respect to $3^{m_1}w3^{m_2}$), where $\max P \leqslant |3^{m_1}w|$ and the smallest m_1 elements of P are $1, 2, \ldots, m_1$, and $\min Q \geqslant m_1 + 1$ and the largest m_2 elements of Q are $|3^{m_1}w| + 1, |3^{m_1}w| + 2, \ldots, |3^{m_1}w3^{m_2}|$, and suppose that Q is offset to the right of P.

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$$\begin{array}{cc|cccccccccccccccccccccc|cc}
\underline{3} & \underline{3} & \underline{\overline{3}} & \underline{\overline{3}} & \underline{\overline{0}} & \underline{\overline{3}} & \underline{\overline{0}} & \underline{\overline{3}} & \underline{\overline{3}} & \underline{\overline{0}} & \underline{\overline{3}} & \underline{\overline{0}} & \underline{\overline{3}} & \overline{3} & \underline{\overline{3}} & \underline{\overline{0}} & \underline{\overline{3}} & \underline{\overline{0}} & \underline{\overline{3}} & \underline{3} & \underline{\overline{3}} & \overline{0} & \underline{\overline{3}} & \overline{0} & \overline{3} & \overline{3} \\
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		j_0																	j_0'						
$\underline{3}$	$\underline{3}$	$\overline{\underline{3}}$	$\overline{\underline{3}}$	$\overline{\underline{0}}$	$\overline{\underline{3}}$	$\overline{\underline{0}}$	$\overline{\underline{3}}$	$\overline{\underline{3}}$	$\overline{\underline{0}}$	$\overline{\underline{3}}$	$\overline{\underline{0}}$	$\overline{\underline{3}}$	$\overline{3}$	$\overline{\underline{3}}$	$\overline{\underline{0}}$	$\overline{\underline{3}}$	$\overline{\underline{0}}$	$\overline{\underline{3}}$	$\underline{3}$	$\overline{\underline{3}}$	$\overline{0}$	$\overline{\underline{3}}$	$\overline{0}$	$\overline{3}$	$\overline{3}$
1	2	3	🐱$_q$	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	🐱$_p$	22	23	24	25	26

Setting the stage

We write $z = 3^{m_1} w 3^{m_2}$. Define the values $j_0, j_0', j_1, j_1', \ldots$ as follows:

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		j_0																$\overleftarrow{\jmath_0'}$	j_0'						
$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{3}$
1	2	3	🐱$_q$	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	🐱$_p$	22	23	24	25	26

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We write $z = 3^{m_1} w 3^{m_2}$. Define the values $j_0, j'_0, j_1, j'_1, \ldots$ as follows:

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$$
\begin{array}{cc|cccccccccccccccccccccc|cc}
 & & j_0 & & & & & & j_1 & & & & & & & & & & \overleftarrow{\jmath'_0} & j'_0 & & & & & & \\
\underline{3} & \underline{3} & \underline{\overline{3}} & \underline{\overline{3}} & \underline{\overline{0}} & \underline{\overline{3}} & \underline{\overline{0}} & \underline{\overline{3}} & \underline{\overline{3}} & \underline{\overline{0}} & \underline{\overline{3}} & \underline{\overline{0}} & \underline{\overline{3}} & \overline{3} & \underline{\overline{3}} & \underline{\overline{0}} & \underline{\overline{3}} & \underline{\overline{0}} & \underline{\overline{3}} & \underline{3} & \underline{\overline{3}} & \overline{0} & \underline{\overline{3}} & \overline{0} & \overline{3} & \overline{3} \\
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		j_0						j_1				j_1'						$\overleftarrow{\jmath_0'}$	j_0'						
3	3	3	3	0	3	0	3	3	0	3	0	3	3	3	0	3	0	3	3	3	0	3	0	3	3
1	2	3	$_q$	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	$_p$	22	23	24	25	26

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		j_0						j_1				j_1'		j_2				$\overleftarrow{\jmath_0'}$	j_0'						
$\underline{3}$	$\underline{3}$	$\overline{\underline{3}}$	$\overline{\underline{3}}$	$\overline{\underline{0}}$	$\overline{\underline{3}}$	$\overline{\underline{0}}$	$\overline{\underline{3}}$	$\overline{\underline{3}}$	$\overline{\underline{0}}$	$\overline{\underline{3}}$	$\overline{\underline{0}}$	$\overline{\underline{3}}$	$\overline{3}$	$\overline{\underline{3}}$	$\overline{\underline{0}}$	$\overline{\underline{3}}$	$\overline{\underline{0}}$	$\overline{\underline{3}}$	$\underline{3}$	$\overline{\underline{3}}$	$\overline{0}$	$\overline{\underline{3}}$	$\overline{0}$	$\overline{3}$	$\overline{3}$
1	2	3	🐱$_q$	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	🐱$_p$	22	23	24	25	26

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		j_0					j_2'	j_1				j_1'		j_2				$\overleftarrow{\jmath_0'}$	j_0'						
$\underline{3}$	$\underline{3}$	$\overline{\underline{3}}$	$\overline{\underline{3}}$	$\overline{\underline{0}}$	$\overline{\underline{3}}$	$\overline{\underline{0}}$	$\overline{\underline{3}}$	$\overline{\underline{3}}$	$\overline{\underline{0}}$	$\overline{\underline{3}}$	$\overline{\underline{0}}$	$\overline{\underline{3}}$	$\overline{3}$	$\overline{\underline{3}}$	$\overline{\underline{0}}$	$\overline{\underline{3}}$	$\overline{\underline{0}}$	$\overline{\underline{3}}$	$\underline{3}$	$\overline{\underline{3}}$	$\overline{0}$	$\overline{\underline{3}}$	$\overline{0}$	$\overline{3}$	$\overline{3}$
1	2	3	🐱$_q$	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	🐱$_p$	22	23	24	25	26

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		j_0					j_2'	j_1				j_1'		j_2				$\overleftarrow{\jmath_0'}$	j_0'	j_3					
$\underline{3}$	$\underline{3}$	$\overline{\underline{3}}$	$\overline{\underline{3}}$	$\overline{\underline{0}}$	$\overline{\underline{3}}$	$\overline{\underline{0}}$	$\overline{\underline{3}}$	$\overline{\underline{3}}$	$\overline{\underline{0}}$	$\overline{\underline{3}}$	$\overline{\underline{0}}$	$\overline{\underline{3}}$	$\overline{3}$	$\overline{\underline{3}}$	$\overline{\underline{0}}$	$\overline{\underline{3}}$	$\overline{\underline{0}}$	$\overline{\underline{3}}$	$\underline{3}$	$\overline{\underline{3}}$	0	$\overline{\underline{3}}$	0	$\overline{3}$	$\overline{3}$
1	2	3	🐱$_q$	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	🐱$_p$	22	23	24	25	26

Dissection

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Let τ be the largest integer such that j_τ (from the previous procedure) is defined. Then the sequence $x_1, x_2, \ldots, x_{\tau+1}$, with $x_i = z[j_{i-1}, j_i - 1]$ for $1 \leqslant i \leqslant \tau$ and $x_{\tau+1} = z[j_\tau, |3^{m_1} w|]$, is called the dissection of w, and the value τ is called the tally of the dissection (clearly, those notions depend not only on w but also on m_1, m_2, P, Q, but all these parameters will always be clear from the context).

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			j_0					j'_2	j_1				j'_1		j_2				$\overleftarrow{j'_0}$	j'_0	j_3						
$\underline{3}$	$\underline{3}$	$\Vert$	$\overline{\underline{3}}$	$\overline{\underline{3}}$	$\overline{\underline{0}}$	$\overline{\underline{3}}$	$\overline{\underline{0}}$	$\overline{\underline{3}}$	$\overline{\underline{3}}$	$\overline{\underline{0}}$	$\overline{\underline{3}}$	$\overline{\underline{0}}$	$\overline{\underline{3}}$	$\overline{3}$	$\overline{\underline{3}}$	$\overline{\underline{0}}$	$\overline{\underline{3}}$	$\overline{\underline{0}}$	$\overline{\underline{3}}$	$\underline{3}$	$\overline{\underline{3}}$	$\overline{0}$	$\overline{\underline{3}}$	$\overline{0}$	$\Vert$	$\overline{3}$	$\overline{3}$
1	2		3	🐱$_q$	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	🐱$_p$	22	23	24		25	26

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$$
\begin{array}{cc|cccccc|cccccc|cccccc|cccc|cc}
 & & j_0 & & & & & j_2' & j_1 & & & & j_1' & & j_2 & & & & \overleftarrow{j_0'} & j_0' & j_3 & & & & & \\
\underline{3} & \underline{3} & \overline{\underline{3}} & \overline{\underline{3}} & \overline{\underline{0}} & \overline{\underline{3}} & \overline{\underline{0}} & \overline{\underline{3}} & \overline{\underline{3}} & \overline{\underline{0}} & \overline{\underline{3}} & \overline{\underline{0}} & \overline{\underline{3}} & \overline{3} & \overline{\underline{3}} & \overline{\underline{0}} & \overline{\underline{3}} & \overline{\underline{0}} & \overline{\underline{3}} & \underline{3} & \overline{\underline{3}} & \overline{0} & \overline{\underline{3}} & \overline{0} & \overline{3} & \overline{3} \\
1 & 2 & 3 & \text{🐱}_q & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & \text{🐱}_p & 22 & 23 & 24 & 25 & 26
\end{array}
$$

Refined dissection

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We also define the refined dissection of w as follows. Let $j_0^\diamondsuit, j_1^\diamondsuit, j_2^\diamondsuit, \ldots$ be the sequence defined in the same way as $j_0, j_1, j_2, \ldots$ but with $j_0^\diamondsuit = \sigma_Q(|\overleftarrow{3^{m_1} w}|)$ instead of $j_0 = m_1 + 1$. Let $x_i' = z[j_{i-1}, j_{i-1}^\diamondsuit - 1]$ and $x_i'' = z[j_{i-1}^\diamondsuit, j_i - 1]$ for $1 \leqslant i \leqslant \tau$, while $x_{\tau+1}' = z[j_\tau, j_\tau^\diamondsuit - 1]$ and $x_{\tau+1}'' = z[j_\tau^\diamondsuit, |3^{m_1} w|]$ if $j_\tau^\diamondsuit$ is defined, respectively $x_{\tau+1}' = z[j_\tau, |3^{m_1} w|]$ if $j_\tau^\diamondsuit$ is undefined (and $x_{\tau+1}''$ is undefined in this last case). Then the sequence $x_1', x_1'', x_2', x_2'', \ldots$ is the refined dissection of w.

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$$\begin{array}{cc|cccccc|cccccc|cccccc|cccc|cc}
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\underline{3} & \underline{3} & \underline{\overline{3}} & \underline{\overline{3}} & \overline{0} & \underline{\overline{3}} & \overline{0} & \underline{\overline{3}} & \underline{\overline{3}} & \underline{\overline{0}} & \underline{\overline{3}} & \underline{\overline{0}} & \underline{\overline{3}} & \overline{3} & \underline{\overline{3}} & \underline{\overline{0}} & \underline{\overline{3}} & \underline{\overline{0}} & \underline{\overline{3}} & \underline{3} & \underline{\overline{3}} & \overline{0} & \underline{\overline{3}} & \overline{0} & \overline{3} & \overline{3} \\
1 & 2 & 3 & \text{🐱}_q & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & \text{🐱}_p & 22 & 23 & 24 & 25 & 26
\end{array}$$

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$$
\begin{array}{cc|cccccc|cccccc|cccccc|cccc|cc}
 & & j_0 & & j_0^\diamond & & & j_2' & j_1 & j_1^\diamond & & & j_1' & & j_2 & j_2^\diamond & & & \overleftarrow{j_0'} & j_0' & j_3 & j_3^\diamond & & & & \\
\underline{3} & \underline{3} & \overline{\underline{3}} & \overline{\underline{3}} & \overline{\underline{0}} & \overline{\underline{3}} & \overline{\underline{0}} & \overline{\underline{3}} & \overline{\underline{3}} & \overline{\underline{0}} & \overline{\underline{3}} & \overline{\underline{0}} & \overline{\underline{3}} & \overline{3} & \overline{\underline{3}} & \overline{\underline{0}} & \overline{\underline{3}} & \overline{\underline{0}} & \overline{\underline{3}} & \underline{3} & \overline{\underline{3}} & \overline{0} & \overline{\underline{3}} & \overline{0} & \overline{3} & \overline{3} \\
1 & 2 & 3 & 🐱_q & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 🐱_p & 22 & 23 & 24 & 25 & 26
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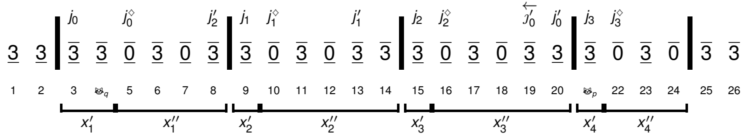

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- On the set $\{1, 2, \ldots, |z|\}$, define:

$$s \sim s' \text{ if there exists } i \in \mathbb{Z} \text{ such that } s' = (\sigma_Q \circ \sigma_P)^i(s).$$

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- If $s \in P/Q$, then the equivalence classes $[s]_\sim$ and $[\sigma_{P/Q}(s)]_\sim$ will be called *twins*.

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- Let $M_1 = [1, m_1]_\mathbb{N}$, $M_2 = [|w| + m_1 + 1, |z|]_\mathbb{N}$, $M = M_1 \cup M_2$, and let

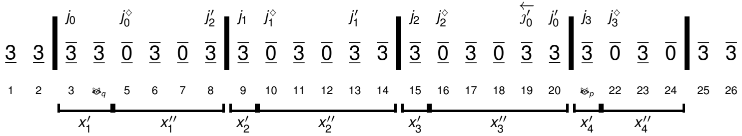
$$M^{☺} = \{s \in M : [s]_\frown \setminus M \subseteq P \cap Q\},$$

with $M_1^{☺}$ and $M_2^{☺}$ similarly defined.

Further preparation

		j_0		$j_0^\diamond$			j_2'	j_1	$j_1^\diamond$			j_1'		j_2	$j_2^\diamond$			$\overleftarrow{j_0'}$	j_0'	j_3	$j_3^\diamond$				
3	3	3	3	0	3	0	3	3	0	3	0	3	3	3	0	3	0	3	3	3	0	3	0	3	3
1	2	3	🐱$_q$	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	🐱$_p$	22	23	24	25	26
		x_1'		x_1''				x_2'	x_2''					x_3'	x_3''					x_4'	x_4''				

Further preparation



The mappings σ_P and σ_Q act as follows:

$$1 \overset{\sigma_P}{\mapsto} 23 \overset{\sigma_Q}{\mapsto} 6 \overset{\sigma_P}{\mapsto} 17 \overset{\sigma_Q}{\mapsto} 11 \overset{\sigma_P}{\mapsto} 11 \overset{\sigma_Q}{\mapsto} 17 \overset{\sigma_P}{\mapsto} 6 \overset{\sigma_Q}{\mapsto} 23 \overset{\sigma_P}{\mapsto} 1;$$

$$2 \overset{\sigma_P}{\mapsto} 21 \overset{\sigma_Q}{\mapsto} 8 \overset{\sigma_P}{\mapsto} 15 \overset{\sigma_Q}{\mapsto} 13 \overset{\sigma_P}{\mapsto} 9 \overset{\sigma_Q}{\mapsto} 19 \overset{\sigma_P}{\mapsto} 4 \overset{\sigma_Q}{\mapsto} 25;$$

$$26 \overset{\sigma_Q}{\mapsto} 3 \overset{\sigma_P}{\mapsto} 20;$$

$$14 \overset{\sigma_Q}{\mapsto} 14;$$

$$24 \overset{\sigma_Q}{\mapsto} 5 \overset{\sigma_P}{\mapsto} 18 \overset{\sigma_Q}{\mapsto} 10 \overset{\sigma_P}{\mapsto} 12 \overset{\sigma_Q}{\mapsto} 16 \overset{\sigma_P}{\mapsto} 7 \overset{\sigma_Q}{\mapsto} 22.$$

Further preparation

$$
\begin{array}{cc|cccccc|cccccc|cccccc|cccc|cc}
 & & j_0 & & j_0^{\diamond} & & & j_2' & j_1 & j_1^{\diamond} & & & j_1' & & j_2 & j_2^{\diamond} & & & \overleftarrow{j_0'} & j_0' & j_3 & j_3^{\diamond} & & & & \\
\underline{3} & \underline{3} & \overline{\underline{3}} & \overline{\underline{3}} & \overline{\underline{0}} & \overline{\underline{3}} & \overline{\underline{0}} & \overline{\underline{3}} & \overline{\underline{3}} & \overline{\underline{0}} & \overline{\underline{3}} & \overline{\underline{0}} & \overline{\underline{3}} & \overline{3} & \overline{\underline{3}} & \overline{\underline{0}} & \overline{\underline{3}} & \overline{\underline{0}} & \overline{\underline{3}} & \underline{3} & \overline{\underline{3}} & \overline{0} & \overline{\underline{3}} & \overline{0} & \overline{3} & \overline{3} \\
1 & 2 & 3 & \text{🐱}_q & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & \text{🐱}_p & 22 & 23 & 24 & 25 & 26 \\
 & & \multicolumn{2}{c}{x_1'} & \multicolumn{4}{c}{x_1''} & x_2' & \multicolumn{5}{c}{x_2''} & x_3' & \multicolumn{5}{c}{x_3''} & x_4' & \multicolumn{3}{c}{x_4''} & &
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$$M^{\text{😄}} = \{1, 2, 25\}.$$

Little gaps

For $a \in \{0, 3\}$, we introduce the following notation:

$$\Xi_a = \bigcup_{X \in \{P, Q\}} \left(\{m_1 + 1 \leqslant s \leqslant |3^{m_1} w| : z[s] = a\} \setminus X \right).$$

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\underline{3} & \underline{3} & \overline{\underline{3}} & \overline{\underline{3}} & \overline{\underline{0}} & \overline{\underline{3}} & \overline{\underline{0}} & \overline{\underline{3}} & \overline{\underline{3}} & \overline{\underline{0}} & \overline{\underline{3}} & \overline{\underline{0}} & \overline{\underline{3}} & \overline{3} & \overline{\underline{3}} & \overline{\underline{0}} & \overline{\underline{3}} & \overline{\underline{0}} & \overline{\underline{3}} & \overline{\underline{3}} & \overline{\underline{3}} & \overline{0} & \overline{\underline{3}} & \overline{0} & \overline{3} & \overline{3} \\
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\end{array}$$

$\Xi_0 = \{22, 24\}$ and $\Xi_3 = \{14, 20\}$.

Little gaps

We have the following bounds on $|\Xi_a|$:

a) $|\Xi_0|$ is greater than or equal to the total number of 0-$\sim$-classes.

b) $|\Xi_3|$ is greater than or equal to the sum of the following two values:
   - the number of 3-$\frown$-classes that contain some element of $M \setminus M^{\smiley}$;
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In fact, we have:

$$|\Xi_0| + |\Xi_3| \geqslant \frac{|w|_3}{2(\tau + 1)}.$$

Lemma 1

Lemma

Let $w \in \{0,3\}^*$, and let m_1, m_2 be nonnegative integers such that $m_1 \leqslant 3^{g(\widetilde{w},0,3)}$ and $m_2 \leqslant 3^{g(w,0,3)}$. Let p and q be subpalindromes of $3^{m_1}w$ and $w3^{m_2}$, respectively. Let P and Q be descriptors of p and q (with respect to $3^{m_1}w3^{m_2}$), where $\max P \leqslant |3^{m_1}w|$ and the smallest m_1 elements of P are $1, 2, \ldots, m_1$, and $\min Q \geqslant m_1 + 1$ and the largest m_2 elements of Q are $|3^{m_1}w| + 1, |3^{m_1}w| + 2, \ldots, |3^{m_1}w3^{m_2}|$, and suppose that Q is offset to the right of P. Then

$$|p| + |q| \leqslant 2|w| + m - \frac{|w|_3}{2(\tau + 1)},$$

where $m = m_1 + m_2$ and τ is the tally of the dissection of w.

Lemma 2

Lemma

Let $w \in \{0,1,2,3\}^*$ where $|w|_0 \leqslant |w|_1 \leqslant |w|_2 \leqslant |w|_3$, and let m_1, m_2 be nonnegative integers such that $m_1 \leqslant 3^{g(\widetilde{w},0,3)}$ and $m_2 \leqslant 3^{g(w,0,3)}$. Let p and q be subpalindromes of $3^{m_1}w$ and $w3^{m_2}$, respectively. Let P and Q be descriptors of p and q (with respect to $3^{m_1}w3^{m_2}$), where $\max P \leqslant |3^{m_1}w|$ and the smallest m_1 elements of P are $1, 2, \ldots, m_1$, and $\min Q \geqslant m_1 + 1$ and the largest m_2 elements of Q are $|3^{m_1}w| + 1, |3^{m_1}w| + 2, \ldots, |3^{m_1}w3^{m_2}|$. Suppose that Q is offset to the right of P. Then, if $u = (3^{m_1}w3^{m_2})[Q_1, P_{|p|}]$ if $Q_1 \leqslant P_{|p|}$, and $u = \varepsilon$ otherwise, we have

$$|p| + |q| \leqslant 4|w|_3 + 2|u|_2 + 4|w|_0.$$

Propositions 4–5

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$1p1$		
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	$\checkmark$	

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