Introduction

Definitions

Some interesting

subpalindromes

The main theore

Definitions

 $n \geqslant 4$ On bounding the

MP-ratio from abo

Ternary case

Oueterpery

The main theorem

The proof

Some notation

Propositions 4—F

Putting everythin

And now there are four: Another brick in the wall of the optimal upper bound on the MP-ratio

Kristina Ago

joint work with Bojan Bašić

Department of Mathematics and Informatics, University of Novi Sad, Serbia

WORDS 2025 Nancy, France June 30, 2025

And now there are four

Kristina Ago

Introduction

Definitions
Some interesting roles of

The main theorem

n-ary cas

Definitions
Well-definedness for

On bounding the MP-ratio from above

Ternary case

case

The main theorem

Propositions 1–2

Some notation

Propositions 4-5

Putting everythin together

roles of

Well-definedness for

MP-ratio from above

 Various measures of the degree of palindromicity of words have been introduced and studied in the literature.

Definitions
Some interesting roles of

The main theore

Definitions
Well-definedness for

On bounding the MP-ratio from abov

Ternary case

Ouatorpary

The main theore

The proof

Some notation

Propositions 4-

Propositions 4-

Putting everythin

- Various measures of the degree of palindromicity of words have been introduced and studied in the literature.
- One such measure, which is the main topic of the present talk, is the so-called MP-ratio.

Definitions
Some interesting roles of subpalindromes

The main theore

Definitions
Well-definedness fo $n \ge 4$ On bounding the MP-ratio from above

Ternary case

Quaternary

The main theor

The proof

Some notation

Propositions 4-

Propositions 4–5

 Various measures of the degree of palindromicity of words have been introduced and studied in the literature.

- One such measure, which is the main topic of the present talk, is the so-called MP-ratio.
- It is a rational number greater than or equal to 1, such that, the larger it is, the more the given word is "palindromic".

Definitions
Some interesting roles of subpalindromes

The main theore

Definitions
Well-definedness for $n \ge 4$ On bounding the MP-ratio from above

Ternary case

case

The proof

Some notation
Proposition 3

Propositions 4–5

Propositions 4–5
Putting everything

- Various measures of the degree of palindromicity of words have been introduced and studied in the literature.
- One such measure, which is the main topic of the present talk, is the so-called MP-ratio.
- It is a rational number greater than or equal to 1, such that, the larger it is, the more the given word is "palindromic".
- The concept was originally defined only for binary words, and its extension to larger alphabets was left as an open problem.

roles of

 $n \ge 4$

On bounding the MP-ratio from above

D:-----

Definitions

roles of

The main theore

n-ary cas

Definitions
Well-definedness
n ≥ 4

On bounding the MP-ratio from above

Ternary case
The main theorer

- 1

The main these

The proof

Propositions 1-

Some notation

Proposition 3

Propositions 4-

• *w* – binary word on {0, 1};

Some interest roles of

subpalindro

THE HIGH LIEU

Ti-ally Cast

Well-definedness for

On bounding the

MP-ratio from above

Ternary case
The main theorem

Quaternary

The main theor

The proof

Some notation

Proposition 3

Propositions 4-

Putting everything

- w binary word on {0, 1};
- w contains a subpalindrome of length $\left| \frac{|w|}{2} \right|$

 $n \geqslant 4$

On bounding the MP-ratio from above

Ternary case
The main theore

Quaternary

The main theore

The proof

Some notation

Propositions 4

Butting quantiti

- *w* binary word on {0, 1};
- w contains a subpalindrome of length $\left| \frac{|w|}{2} \right|$
- w is minimal-palindromic

w – binary word on {0, 1};

- w contains a subpalindrome of length $\left| \frac{|w|}{2} \right|$;
- w is minimal-palindromic \leftrightarrow does not contain a subpalindrome longer than $\left\lceil \frac{|w|}{2} \right\rceil$;

On bounding the MP-ratio from above

Ternary case

Ouatorpary

The main theo

The proof

Some notation

Propositions 4...

Putting everythin

- w binary word on {0, 1};
- w contains a subpalindrome of length $\left| \frac{|w|}{2} \right|$;
- w is minimal-palindromic \leftrightarrow does not contain a subpalindrome longer than $\left\lceil \frac{|w|}{2} \right\rceil$;
- $r, s \in \{0, 1\}^*, (r, s)$ is an MP-extension of w

Some notation

Propositions 4-4

Putting everythin

- *w* binary word on {0, 1};
- w contains a subpalindrome of length $\left| \frac{|w|}{2} \right|$;
- w is minimal-palindromic \leftrightarrow does not contain a subpalindrome longer than $\left\lceil \frac{|w|}{2} \right\rceil$;
- $r, s \in \{0, 1\}^*$, (r, s) is an MP-extension of $w \leftrightarrow rws$ is minimal-palindromic;

Propositions 1-Some notation

Propositions 4–6

Putting everything

- *w* binary word on {0, 1};
- w contains a subpalindrome of length $\left| \frac{|w|}{2} \right|$;
- w is minimal-palindromic \leftrightarrow does not contain a subpalindrome longer than $\left\lceil \frac{|w|}{2} \right\rceil$;
- $r, s \in \{0, 1\}^*$, (r, s) is an MP-extension of $w \leftrightarrow rws$ is minimal-palindromic;
- (r, s) is an SMP-extension

Proposition 3

Propositions 4–5

- *w* binary word on {0, 1};
- w contains a subpalindrome of length $\lceil \frac{|w|}{2} \rceil$;
- w is minimal-palindromic \leftrightarrow does not contain a subpalindrome longer than $\left\lceil \frac{|w|}{2} \right\rceil$;
- $r, s \in \{0, 1\}^*$, (r, s) is an MP-extension of $w \leftrightarrow rws$ is minimal-palindromic;
- (r, s) is an SMP-extension $\leftrightarrow |r| + |s|$ is the least possible;

- w binary word on {0, 1};
- w contains a subpalindrome of length $\left| \frac{|w|}{2} \right|$;
- w is minimal-palindromic \leftrightarrow does not contain a subpalindrome longer than $\left\lceil \frac{|w|}{2} \right\rceil$;
- $r, s \in \{0, 1\}^*, (r, s)$ is an MP-extension of $w \leftrightarrow rws$ is minimal-palindromic;
- (r, s) is an SMP-extension $\leftrightarrow |r| + |s|$ is the least possible;
- MP-ratio of the word w:

Propositions 1–2 Some notation Proposition 3 Propositions 4–5 Putting everythin

- *w* binary word on {0, 1};
- w contains a subpalindrome of length $\lceil \frac{|w|}{2} \rceil$;
- w is minimal-palindromic \leftrightarrow does not contain a subpalindrome longer than $\left\lceil \frac{|w|}{2} \right\rceil$;
- $r, s \in \{0, 1\}^*$, (r, s) is an MP-extension of $w \leftrightarrow rws$ is minimal-palindromic;
- (r, s) is an SMP-extension $\leftrightarrow |r| + |s|$ is the least possible;
- MP-ratio of the word w: $\frac{|rws|}{|w|}$, (r, s) SMP-extension of w.

Introduction

D:-----

Definitions

Some interesting roles of subpalindromes

The main theorem

THE HAIT GIEOR

Definition

Well-definedness for $n \ge 4$

On bounding the

MP-ratio from above

Ternary case
The main theorer

Oueterner

The main theory

The proof

Propositions 1-

Describing 0

Propositions 4

Putting everythin

Some interesting roles of subpalindromes

Introduction

Binary cas

Definitions

Some interesting

subpalindromes

The main theor

Definitions
Well-definedness

On bounding the

Ternary case

The main theore

case

The proof

Some notation

Proposition 3

Propositions 4-

Putting everything

Some interesting roles of subpalindromes

Theorem (Holub, Saari; 2009)

Every binary minimal-palindromic word is abelian unbordered.

Some interesting roles of subpalindromes

The main theor

n-ary case
Definitions

Well-definedness for $n \geqslant 4$

On bounding the MP-ratio from above

Ternary case

The main theore

The main theore

The proof

Some notation

Proposition 3
Propositions 4–8

Putting everything together

Some interesting roles of subpalindromes

Theorem (Holub, Saari; 2009)

Every binary minimal-palindromic word is abelian unbordered.

Theorem (Holub, Saari; 2009)

Every binary word is characterized, up to reversal, by the set of its subpalindromes.

Introduction

Introduction

Dofinitio

Some interesti

subpalindrom

The main theorem

n-ary cas

Definitions

Well-definedness for $n \ge 4$

On bounding the MP-ratio from above

Ternary case

The main theore

Quaternary

The main theore

The proof

Propositions 1-

Some notation

- rioposition

Propositions 4–

The optimal upper bound on the MP-ratio

Introduction

Dinami

Some interesting roles of

The main theorem

Definitions
Well-definedness

m ≥ 4

On bounding the MP-ratio from abo

Ternary case

The main theor

case

The main theon

Propositions 1-

Proposition 3

Propositions 4-

Putting everythin

The optimal upper bound on the MP-ratio

Theorem (Holub, Saari; 2009)

The MP-ratio of any binary word is at most 4.

Introduction

Definitions
Some interesti

The main theorem

Definitions
Well-definedness f

On bounding the MP-ratio from abo

Ternary cas

The main theo

The main theo

The proof

Some notation

Propositions 4-

Putting everythi

The optimal upper bound on the MP-ratio

Theorem (Holub, Saari; 2009)

The MP-ratio of any binary word is at most 4.

Proof (sketch).

$$0^{|w|+|w|_1}w1^{|w|+|w|_0}$$

Introduction

Definitions
Some interesti

The main theorem

Definitions
Well-definedness

On bounding the MP-ratio from above

Ternary case

The main theor

The main theo

The proof

Some notation

Proposition 3
Propositions 4-

Putting everythin

The optimal upper bound on the MP-ratio

Theorem (Holub, Saari; 2009)

The MP-ratio of any binary word is at most 4.

Proof (sketch).

$$0^{|w|+|w|_1}w1^{|w|+|w|_0}$$

Theorem (Holub, Saari; 2009) This upper bound is the best possible.

Introduction

IIIII Odučtioi

Definition

Some interesting

subpalindromes

n-ary ca

Definitions

Well-definedness $n \ge 4$

On bounding the MP-ratio from above

Ternary case

The main theore

Oustornary

The main theore

The proof

Propositions 1-

Some notation

Froposition

Putting everythi

The main theore

n-ary cas

Definitions

Well-definedness for

On bounding the MP-ratio from above

Ternary case

The main theor

case

The proof

Propositions 1-

Dronocition 2

Propositions 4-

Putting everythin

Definitions

• w - n-ary word on $\{0, 1, ..., n - 1\}$;

Well-definedness

On bounding the

MP-ratio from above

Ternary cas

I ne main theo

The main theore

The proof

Some notation

Bronsoitions 4

Propositions 4-

- w n-ary word on $\{0, 1, ..., n 1\}$;
- w contains a subpalindrome of length $\left\lceil \frac{|w|}{n} \right\rceil$;

Well-definedness

On bounding the

On bounding the MP-ratio from above

Ternary case
The main theorem

Ousternary

The main theo

The proof

Some notation

Dropositions 4

Putting everythi

- w n-ary word on $\{0, 1, ..., n 1\}$;
- w contains a subpalindrome of length $\left\lceil \frac{|w|}{n} \right\rceil$;
- w is minimal-palindromic

- w n-ary word on $\{0, 1, ..., n 1\}$;
- w contains a subpalindrome of length $\left| \frac{|w|}{n} \right|$;
- w is minimal-palindromic \leftrightarrow does not contain a subpalindrome longer than $\left\lceil \frac{|w|}{n} \right\rceil$;

- w n-ary word on $\{0, 1, ..., n 1\}$;
- w contains a subpalindrome of length $\left\lceil \frac{|w|}{n} \right\rceil$;
- w is minimal-palindromic \leftrightarrow does not contain a subpalindrome longer than $\left\lceil \frac{|w|}{n} \right\rceil$;
- $r, s \in \{0, 1, ..., n-1\}^*$, (r, s) is an MP-extension of $w \leftrightarrow rws$ is minimal-palindromic;
- (r, s) is an SMP-extension $\leftrightarrow |r| + |s|$ is the least possible;
- MP-ratio of the word w: $\frac{|rws|}{|w|}$, (r, s) SMP-extension of w.

Propositions 1–2 Some notation Proposition 3 Propositions 4–5 Putting everythin

- w n-ary word on $\{0, 1, ..., n 1\}$;
- w contains a subpalindrome of length $\left\lceil \frac{|w|}{n} \right\rceil$;
- w is minimal-palindromic \leftrightarrow does not contain a subpalindrome longer than $\left\lceil \frac{|w|}{n} \right\rceil$;
- $r, s \in \{0, 1, ..., n-1\}^*$, (r, s) is an MP-extension of $w \leftrightarrow rws$ is minimal-palindromic; Does it always exist???
- (r, s) is an SMP-extension $\leftrightarrow |r| + |s|$ is the least possible;
- MP-ratio of the word w: $\frac{|rws|}{|w|}$, (r, s) SMP-extension of w.

Introduction

Dinami

Definitions

Some interestir roles of

subpalindromes

The main theorer

n-ary cas

Definitions

Well-definedness for $n \ge 4$

On bounding the MP-ratio from above

Ternary case

O------

The main theor

The proof

Propositions 1-

Some notation

- Toposition

Putting everythi

Well-definedness for $n \ge 4$

Introduction

IIIIIOddciio

Definitio

Some interesti

roles of

The main theore

n-ary cas

Definitions

Well-definedness for $n \ge 4$

On bounding the MP-ratio from above

Ternary case

The main theor

The main theore

The proof

Propositions 1-

Proposition 3

Propositions 4-5

Well-definedness for $n \ge 4$

Theorem (A., Bašić; 2021)

Each n-ary word has an MP-extension.

Introduction

Б.

Definitions

Some interesting

subpalindro

The main theo

Definitions

Well-definedness for $n \ge 4$

On bounding the

MP-ratio from above

Ternary case
The main theorem

case

The main theor

Propositions 1-

Proposition 3

Propositions 4-

Putting everythin

Well-definedness for $n \ge 4$

Theorem (A., Bašić; 2021) Each n-ary word has an MP-extension.

Proof (sketch).

Definitions
Well-definedness for

Well-definedness $n \geqslant 4$

On bounding the MP-ratio from abo

Ternary case
The main theore

Quaternary

The main theor

The proof

Some notation

Propositions 4...

Putting everythin

Well-definedness for $n \ge 4$

Theorem (A., Bašić; 2021)

Each n-ary word has an MP-extension.

Proof (sketch).

Let $w \in \{0, 1, \dots, n-1\}^*$, $n \geqslant 4$.

Well-definedness for $n \ge 4$

Theorem (A., Bašić; 2021)

Each n-ary word has an MP-extension.

Proof (sketch).

Let $w \in \{0, 1, ..., n-1\}^*, n \geqslant 4$.

Let $M = 2^{\lceil \frac{n}{2} \rceil} + 2^{\lfloor \frac{n}{2} \rfloor - 1} - 3$; exceptionally, if n = 4 or n = 5, we

define M = 4, respectively M = 8 instead.

Well-definedness for $n \ge 4$

Theorem (A., Bašić; 2021)

Each n-ary word has an MP-extension.

Proof (sketch).

Let $w \in \{0, 1, ..., n-1\}^*, n \geqslant 4$.

Let $M = 2^{\lceil \frac{n}{2} \rceil} + 2^{\lfloor \frac{n}{2} \rfloor - 1} - 3$; exceptionally, if n = 4 or n = 5, we define M = 4, respectively M = 8 instead. Let:

$$r = 0^{l_0} 1^{l_1} \dots (n-2)^{l_{n-2}}, \ \ s = 1^{r_1} 2^{r_2} \dots (n-1)^{r_{n-1}},$$

The proof

Propositions 1-Some notation

Proposition 3 Propositions 4–5

Propositions 4–5
Putting everythin

Well-definedness for $n \ge 4$

Theorem (A., Bašić; 2021)

Each n-ary word has an MP-extension.

Proof (sketch).

Let $w \in \{0, 1, \dots, n-1\}^*$, $n \geqslant 4$.

Let $M = 2^{\lceil \frac{n}{2} \rceil} + 2^{\lfloor \frac{n}{2} \rfloor - 1} - 3$; exceptionally, if n = 4 or n = 5, we define M = 4, respectively M = 8 instead. Let:

$$r = 0^{l_0} 1^{l_1} \dots (n-2)^{l_{n-2}}, \ \ s = 1^{r_1} 2^{r_2} \dots (n-1)^{r_{n-1}},$$

where $I_0 = M|w| - |w|_0$ and

$$I_i = \left\{ \begin{array}{ll} (M-2^i+1)|w|, & \text{for } i=1,\ldots,\lceil\frac{n}{2}\rceil-1; \\ (2^{n-1-i}-1)|w|-|w|_i, & \text{for } i=\lceil\frac{n}{2}\rceil,\ldots,n-2, \end{array} \right.$$

while
$$r_{n-1} = M|w| - |w|_{n-1}$$
 and

$$r_i = \left\{ \begin{array}{ll} (2^i - 1)|w| - |w|_i, & \text{for } i = 1, \dots, \lceil \frac{n}{2} \rceil - 1; \\ (M - 2^{n - 1 - i} + 1)|w|, & \text{for } i = \lceil \frac{n}{2} \rceil, \dots, n - 2. \end{array} \right.$$

together

Well-definedness for $n \ge 4$

Theorem (A., Bašić; 2021)

Each n-ary word has an MP-extension.

Proof (sketch).

Let $w \in \{0, 1, \dots, n-1\}^*$, $n \geqslant 4$.

Let $M = 2^{\lceil \frac{n}{2} \rceil} + 2^{\lfloor \frac{n}{2} \rfloor - 1} - 3$; exceptionally, if n = 4 or n = 5, we define M = 4, respectively M = 8 instead. Let:

$$r = 0^{l_0} 1^{l_1} \dots (n-2)^{l_{n-2}}, \ \ s = 1^{r_1} 2^{r_2} \dots (n-1)^{r_{n-1}},$$

where $I_0 = M|w| - |w|_0$ and

$$I_i = \left\{ \begin{array}{ll} (M-2^i+1)|w|, & \text{for } i=1,\ldots,\lceil\frac{n}{2}\rceil-1; \\ (2^{n-1-i}-1)|w|-|w|_i, & \text{for } i=\lceil\frac{n}{2}\rceil,\ldots,n-2, \end{array} \right.$$

while $r_{n-1} = M|w| - |w|_{n-1}$ and

$$r_i = \begin{cases} (2^i - 1)|w| - |w|_i, & \text{for } i = 1, \dots, \lceil \frac{n}{2} \rceil - 1; \\ (M - 2^{n-1-i} + 1)|w|, & \text{for } i = \lceil \frac{n}{2} \rceil, \dots, n - 2. \end{cases}$$

Then (r, s) is an MP-extension of w.



Introduction

IIIII Odučtioi

Definition

Some interesting

subpalindromes

The main theore

n-ary cas

77 di y 0d

Well-definedness for $n \ge 4$

On bounding the MP-ratio from above

Ternary cas

The main theo

case

Th-----

Trie proof

Propositions I-

D ''' o

. .

Putting quanth

On bounding the MP-ratio from above

On bounding the MP-ratio from above

Theorem (A., Bašić; 2021)

Let w be an n-ary word for $n \ge 4$, and let

$$M = \left\{ \begin{array}{ll} 4, & \text{if } n = 4; \\ 8, & \text{if } n = 5; \\ 2^{\left \lceil \frac{n}{2} \right \rceil} + 2^{\left \lfloor \frac{n}{2} \right \rfloor - 1} - 3, & \text{if } n \geqslant 6. \end{array} \right.$$

Then the MP-ratio of w is not greater than nM.

Some notation

Proposition 3
Propositions 4—

Propositions 4-

On bounding the MP-ratio from above

Theorem (A., Bašić; 2021)

Let w be an n-ary word for $n \ge 4$, and let

$$M = \begin{cases} 4, & \text{if } n = 4; \\ 8, & \text{if } n = 5; \\ 2^{\lceil \frac{n}{2} \rceil} + 2^{\lfloor \frac{n}{2} \rfloor - 1} - 3, & \text{if } n \geqslant 6. \end{cases}$$

Then the MP-ratio of w is not greater than nM.

Theorem (A., Bašić; 2021)

The optimal upper bound on the MP-ratio is not less than 2n.

Introduction

_.

Definitions

Some interestir

subpalindromes

The main theore

n-arv cas

Traily Ca

Well-definedness for n ≥ 4

On bounding the MP-ratio from above

Ternary case

The main theo

case

The main theor

The proof

Propositions 1-

Some notation

. .

Putting everyth

On optimality

Some interestin roles of subpalindromes

The main theo

Definitions
Well-definedness for

On bounding the MP-ratio from above

Ternary cas

The main theo

The main theore

The proof

Some notation Proposition 3

Propositions 4-

Putting everythin

On optimality

Now we know that the optimal upper bound on the MP-ratio for $n \ge 4$ is somewhere between

$$2n$$
 and $\sim 2^{\frac{n}{2}}n$,

and it remains an open problem to narrow (or even better, close) this gap.

Introduction

Definitions

Some interest

subpalindromes

The main theore

n-ary cas

many out

Well-definedness

n ≥ 4

MP-ratio from above

Ternary case
The main theorem

The main theore

Qualerria

The main theor

The proof

Propositions 1-

Some notation

Putting everyth

The main theorem

roles of

Ternary case The main theorem

The main theorem

Theorem (A., Bašić; 2021)

The MP-ratio is well-defined in the ternary case, it is bounded from above by 6 and this upper bound is the best possible.

Ternary case The main theorem

The main theorem

Theorem (A., Bašić; 2021)

The MP-ratio is well-defined in the ternary case, it is bounded from above by 6 and this upper bound is the best possible.

Proof (sketch).

$$f(w) = 0^{2|w|-|w|_0} 2^{2|w|-|w|_2-g'(w,0,2)} \ w \ 2^{g'(w,0,2)} 1^{2|w|-|w|_1}$$

$$f'(w) = 1^{2|w|-|w|_1} 2^{g'(\widetilde{w},0,2)} \ w \ 2^{2|w|-|w|_2-g'(\widetilde{w},0,2)} 0^{2|w|-|w|_0}$$

Introductio

Definitions
Some interestin

roles of subpalindromes The main theore

Definitions
Well-definedness for ≥ 4

On bounding the MP-ratio from abo

Ternary case
The main theorem

case

The main theor

Propositions 1-Some notation

Propositions 4

Propositions 4-5

The main theorem

Theorem (A., Bašić; 2021)

The MP-ratio is well-defined in the ternary case, it is bounded from above by 6 and this upper bound is the best possible.

Proof (sketch).

$$f(w) = 0^{2|w|-|w|_0} 2^{2|w|-|w|_2-g'(w,0,2)} w 2^{g'(w,0,2)} 1^{2|w|-|w|_1}$$

$$f'(w) = 1^{2|w|-|w|_1} 2^{g'(\widetilde{w},0,2)} \ w \ 2^{2|w|-|w|_2-g'(\widetilde{w},0,2)} 0^{2|w|-|w|_0}$$

$$g'(w, a, b) = \max (\{2|w[i, |w|]|_a - |w[i, |w|]|_b : i = 1, 2, \dots, j(a, w)\} \cup \{0\}),$$

where $j(a, w)$ denotes the position of the last occurrence of a in w , and $j(a, w) = 0$ if a does not occur in w .

Introduction

Definitions
Some interesting

Some interesting roles of subpalindromes The main theore

Definitions
Well-definedness to $n \ge 4$ On bounding the

Ternary case

The main theorem

Case
The main theore

The proof
Propositions 1-

Propositions 4-4

Propositions 4–5

Putting every

The main theorem

Theorem (A., Bašić; 2021)

The MP-ratio is well-defined in the ternary case, it is bounded from above by 6 and this upper bound is the best possible.

Proof (sketch).

$$f(w) = 0^{2|w|-|w|_0} 2^{2|w|-|w|_2-g'(w,0,2)} w 2^{g'(w,0,2)} 1^{2|w|-|w|_1}$$

$$f'(w) = 1^{2|w|-|w|_1} 2^{g'(\widetilde{w},0,2)} \ w \ 2^{2|w|-|w|_2-g'(\widetilde{w},0,2)} 0^{2|w|-|w|_0}$$

$$g'(w, a, b) = \max \left(\left\{ 2|w[i, |w|] \right|_a - \left| w[i, |w|] \right|_b : i = 1, 2, \dots, j(a, w) \right\} \cup \{0\} \right),$$
 where $j(a, w)$ denotes the position of the last occurrence of a in w , and $j(a, w) = 0$ if a does not occur in w .

Theorem (A., Bašić; 2022)

Let $u \in \{1,2\}^*$, $t,v \in 2^*$ and let p and q be subpalindromes of tu and uv, respectively. If |p|+|q|>2|u|,

The main theorem

Theorem (A., Bašić; 2021)

The MP-ratio is well-defined in the ternary case, it is bounded from above by 6 and this upper bound is the best possible.

Proof (sketch).

$$f(w) = 0^{2|w| - |w|_0} 2^{2|w| - |w|_2 - g'(w, 0, 2)} w 2^{g'(w, 0, 2)} 1^{2|w| - |w|_1}$$

$$f'(w) = 1^{2|w|-|w|_1} 2^{g'(\widetilde{w},0,2)} \ w \ 2^{2|w|-|w|_2-g'(\widetilde{w},0,2)} 0^{2|w|-|w|_0}$$

$$g'(w, a, b) = \max (\{2 |w[i, |w|]|_a - |w[i, |w|]|_b : i = 1, 2, \dots, j(a, w)\} \cup \{0\}),$$
 where $j(a, w)$ denotes the position of the last occurence of a in w , and $j(a, w) = 0$

where j(a, w) denotes the position of the last occurrence of a in w, and j(a, w) = 0if a does not occur in w.

Theorem (A., Bašić; 2022)

Let $u \in \{1,2\}^*$, $t,v \in 2^*$ and let p and q be subpalindromes of tu and uv, respectively. If |p| + |q| > 2|u|, then

$$|u|_1\leqslant \frac{|tv|-1}{|tv|}|tuv|_2.$$

Introduction

Binary cas

Some interesting roles of

subpalindromes
The main theorer

ary cas

Definitions
Well-definedness for $n \ge 4$

On bounding the MP-ratio from above

Ternary case
The main theorem

The main theorem Quaternary

The main theoren

The proof

case

Propositions 1-

Some notation

Bronositions

P ...

Putting everythi

And now there are four.

Introduction

_.

Definitions

Some interesting roles of

The main theorem

n-arv cas

Definitions

Well-definedness for n ≥ 4

On bounding the MP-ratio from above

Ternary case

Oueteman neore

The main theorem

The pro

Propositions 1

Some notation

Propositions 4

Putting everythi

Quaternary case

Definitions

Some interesti

subpalindrome

The main th

Definitions

Well-definedness for

On bounding the

MP-ratio from above

The main theorer

Quaternary

The main theorem

Propositions 1

Some notation Proposition 3

Propositions 4-

Putting everythin

Quaternary case

Theorem (A., Bašić; 2025+)

The MP-ratio of any 4-ary word is at most 8.

roles of

Well-definedness for

MP-ratio from above

The main theorem

Quaternary case

Theorem (A., Bašić; 2025+)

The MP-ratio of any 4-ary word is at most 8.

Proof (sketch).

Introduction

Definitions Some interestin

Some interesting roles of subpalindromes

The main theor

n-ary case
Definitions
Well-definedness
n ≥ 4

On bounding the MP-ratio from above

Ternary case

The main theore

The main theorem

Propositions 1
Some notation

Propositions 4-

Propositions 4-

Quaternary case

Theorem (A., Bašić; 2025+)

The MP-ratio of any 4-ary word is at most 8.

Proof (sketch).

We define two extensions of w, and show that at least one of them is an MP-extension.

The main theorem

Quaternary case

Theorem (A., Bašić; 2025+)

The MP-ratio of any 4-ary word is at most 8.

Proof (sketch).

We define two extensions of w, and show that at least one of them is an MP-extension.

subpalindr The main t

n-ary case
Definitions
Well-definedness for n ≥ 4

On bounding the MP-ratio from abo

Ternary case
The main theore

Quaternary

The main theorem

Propositions 1

Proposition 3
Propositions 4-

Propositions 4–5
Putting everythin

Quaternary case

Theorem (A., Bašić; 2025+)

The MP-ratio of any 4-ary word is at most 8.

Proof (sketch).

We define two extensions of w, and show that at least one of them is an MP-extension.

Let us call them $f_1(w)$ and $f_2(w)$, respectively.

On bounding the MP-ratio from abo

Ternary case
The main theore

Quaternary

The main theorem

Propositions 1–2 Some notation Proposition 3 Propositions 4–5 Quaternary case

Theorem (A., Bašić; 2025+)

The MP-ratio of any 4-ary word is at most 8.

Proof (sketch).

We define two extensions of w, and show that at least one of them is an MP-extension.

Let us call them $f_1(w)$ and $f_2(w)$, respectively. We have $|f_1(w)| = |f_2(w)| = 8|w|$, and we prove that at least one of the words $f_1(w)$ and $f_2(w)$ does not have a subpalindrome whose length exceeds 2|w|. subpalindre The main t

Definitions
Well-definedness f $n \ge 4$ On bounding the

Ternary case

Quaternary

The main theorem

Propositions 1–2
Some notation
Proposition 3
Propositions 4–5
Putting everythin

Quaternary case

Theorem (A., Bašić; 2025+)

The MP-ratio of any 4-ary word is at most 8.

Proof (sketch).

We define two extensions of w, and show that at least one of them is an MP-extension.

Let us call them $f_1(w)$ and $f_2(w)$, respectively. We have $|f_1(w)| = |f_2(w)| = 8|w|$, and we prove that at least one of the words $f_1(w)$ and $f_2(w)$ does not have a subpalindrome whose length exceeds 2|w|. This will follow by Propositions 1–5.

Introduction

IIIIIoductioi

Definitions

Como intere

roles of

subpalindromes

n ary oad

n-ary cas

Definitions

 $n \geqslant 4$

On bounding the MP-ratio from above

Ternary cas

_

The main theory

Propositions 1–2

Some notation

Proposition 3

Propositions 4

Putting everythin

Propositions 1–2

Propositions 1-2

Propositions 1–2

$$0^{2|w|-|w|_0}3^{2|w|-|w|_3-g(w,0,3)}2^{g(\widetilde{w},1,2)} \quad w \quad 3^{g(w,0,3)}2^{2|w|-|w|_2-g(\widetilde{w},1,2)}1^{2|w|-|w|_1};$$

$$1^{2|w|-|w|_1}2^{2|w|-|w|_2-g(w,1,2)}3^{g(\widetilde{w},0,3)} \quad w \quad 2^{g(w,1,2)}3^{2|w|-|w|_3-g(\widetilde{w},0,3)}0^{2|w|-|w|_0}.$$

On bounding the MP-ratio from above

Ternary cas

The main theor

The main theo

Propositions 1–2

Some notation

Propositions 4-

Putting everythin

Propositions 1–2

$$0^{2|w|-|w|_0} 3^{2|w|-|w|_3-g(w,0,3)} 2^{g(\widetilde{w},1,2)} \ \, w \ \, 3^{g(w,0,3)} 2^{2|w|-|w|_2-g(\widetilde{w},1,2)} 1^{2|w|-|w|_1}; \\ 1^{2|w|-|w|_1} 2^{2|w|-|w|_2-g(w,1,2)} 3^{g(\widetilde{w},0,3)} \ \, w \ \, 2^{g(w,1,2)} 3^{2|w|-|w|_3-g(\widetilde{w},0,3)} 0^{2|w|-|w|_0}.$$

Proposition

The length of an arbitrary subpalindrome of the form 0p0 in each of the words $f_1(w)$ and $f_2(w)$ is less than or equal to 2|w|.

Some notation Proposition 3

Propositions 4-

Putting everythin

Propositions 1–2

$$0^{2|w|-|w|_0} 3^{2|w|-|w|_3-g(w,0,3)} 2^{g(\widetilde{w},1,2)} \quad w \quad 3^{g(w,0,3)} 2^{2|w|-|w|_2-g(\widetilde{w},1,2)} 1^{2|w|-|w|_1};$$

$$1^{2|w|-|w|_1} 2^{2|w|-|w|_2-g(w,1,2)} 3^{g(\widetilde{w},0,3)} \quad w \quad 2^{g(w,1,2)} 3^{2|w|-|w|_3-g(\widetilde{w},0,3)} 0^{2|w|-|w|_0}.$$

Proposition

The length of an arbitrary subpalindrome of the form 0p0 in each of the words $f_1(w)$ and $f_2(w)$ is less than or equal to 2|w|.

Proof (sketch).

Case analysis for $f_1(w)$,

Propositions 4-

Putting everythin

Propositions 1–2

$$0^{2|w|-|w|_0} 3^{2|w|-|w|_3-g(w,0,3)} 2^{g(\widetilde{w},1,2)} \ \, w \ \, 3^{g(w,0,3)} 2^{2|w|-|w|_2-g(\widetilde{w},1,2)} 1^{2|w|-|w|_1}; \\ 1^{2|w|-|w|_1} 2^{2|w|-|w|_2-g(w,1,2)} 3^{g(\widetilde{w},0,3)} \ \, w \ \, 2^{g(w,1,2)} 3^{2|w|-|w|_3-g(\widetilde{w},0,3)} 0^{2|w|-|w|_0}.$$

Proposition

The length of an arbitrary subpalindrome of the form 0p0 in each of the words $f_1(w)$ and $f_2(w)$ is less than or equal to 2|w|.

Proof (sketch).

Case analysis for $f_1(w)$, and for $f_2(w)$ we use that $f_2(w) = \widehat{f_1(\widetilde{w})}$.

Propositions 4-

Propositions 1–2

$$0^{2|w|-|w|_0} 3^{2|w|-|w|_3-g(w,0,3)} 2^{g(\widetilde{w},1,2)} \ \ w \ \ 3^{g(w,0,3)} 2^{2|w|-|w|_2-g(\widetilde{w},1,2)} 1^{2|w|-|w|_1};$$

$$1^{2|w|-|w|_1} 2^{2|w|-|w|_2-g(w,1,2)} 3^{g(\widetilde{w},0,3)} \ \ w \ \ 2^{g(w,1,2)} 3^{2|w|-|w|_3-g(\widetilde{w},0,3)} 0^{2|w|-|w|_0}.$$

Proposition

The length of an arbitrary subpalindrome of the form 0p0 in each of the words $f_1(w)$ and $f_2(w)$ is less than or equal to 2|w|.

Proof (sketch).

Case analysis for $f_1(w)$, and for $f_2(w)$ we use that $f_2(w) = f_1(\widetilde{w})$.

Proposition

The length of an arbitrary subpalindrome of the form 1p1 in each of the words $f_1(w)$ and $f_2(w)$ is less than or equal to 2|w|.

Propositions 1–2

$$0^{2|w|-|w|_0} 3^{2|w|-|w|_3-g(w,0,3)} 2^{g(\widetilde{w},1,2)} \ \ w \ \ 3^{g(w,0,3)} 2^{2|w|-|w|_2-g(\widetilde{w},1,2)} 1^{2|w|-|w|_1};$$

$$1^{2|w|-|w|_1} 2^{2|w|-|w|_2-g(w,1,2)} 3^{g(\widetilde{w},0,3)} \ \ w \ \ 2^{g(w,1,2)} 3^{2|w|-|w|_3-g(\widetilde{w},0,3)} 0^{2|w|-|w|_0}.$$

Proposition

The length of an arbitrary subpalindrome of the form 0p0 in each of the words $f_1(w)$ and $f_2(w)$ is less than or equal to 2|w|.

Proof (sketch).

Case analysis for $f_1(w)$, and for $f_2(w)$ we use that $f_2(w) = f_1(\widetilde{w})$.

Proposition

The length of an arbitrary subpalindrome of the form 1p1 in each of the words $f_1(w)$ and $f_2(w)$ is less than or equal to 2|w|.

Proof.

Analogous to the previous one (where 0s and 1s have switched roles, as well as 2s and 3s, and as well as $f_1(w)$ and $f_2(w)$).

Introduction

Definitions

Some interes

roles of

The main theore

n_arv cas

II-ally Cas

Definitions

n ≥ 4

On bounding the MP-ratio from above

Ternary case

THE HIAH (HEO

case

The main meet

Propositions 1-

Some notation

Proposition

Putting everythi

Some notation

roles of

Well-definedness for $n \ge 4$

MP-ratio from above

Some notation

Some notation

Definition

Let p be a subpalindrome of a word z.

Definitions
Well-definedness for

On bounding the MP-ratio from above

Ternary case

The main theo

The main theo

The proof

Some notation

Proposition 3

Propositions 4

Putting everythin

Some notation

Definition

Let p be a subpalindrome of a word z.

• A descriptor of p (with respect to z) is the set P defined by $P = \{P_1, P_2, \dots, P_{|p|}\}$ where $P_1 < P_2 < \dots < P_{|p|}$ and $p = z[P_1]z[P_2]\dots z[P_{|p|}]$. If a is a letter, then $|P|_a$ denotes the value $|\{P_i \in P : z[P_i] = a\}|$.

Some notation Proposition 3

Proposition 3
Propositions 4-

Putting everythin

Some notation

Definition

Let p be a subpalindrome of a word z.

- A descriptor of p (with respect to z) is the set P defined by $P = \{P_1, P_2, \dots, P_{|p|}\}$ where $P_1 < P_2 < \dots < P_{|p|}$ and $p = z[P_1]z[P_2]\dots z[P_{|p|}]$. If a is a letter, then $|P|_a$ denotes the value $|\{P_i \in P : z[P_i] = a\}|$.
- The mapping σ_P : P → P, defined by σ_P : P_s → P_{|p|-s+1} is called mirroring with respect to P.

Ternary case
The main theore

Case
The main theore

The proof
Propositions 1Some notation

Proposition 3
Propositions 4
Putting everyth

Some notation

Definition

Let p be a subpalindrome of a word z.

- A descriptor of p (with respect to z) is the set P defined by $P = \{P_1, P_2, \ldots, P_{|p|}\}$ where $P_1 < P_2 < \cdots < P_{|p|}$ and $p = z[P_1]z[P_2]\ldots z[P_{|p|}]$. If a is a letter, then $|P|_a$ denotes the value $|\{P_i \in P : z[P_i] = a\}|$.
- The mapping σ_P : P → P, defined by σ_P : P_s → P_{|p|-s+1} is called mirroring with respect to P.

Definition

Let p and q be subpalindromes of some word z, and let P and Q be their descriptors. We say that Q is offset to the right of P if and only if for each s, s' such that $s \in P$, $s' \in Q$, $s' \leq s$, we have $\sigma_Q(s') > \sigma_P(s)$.

On bounding the MP-ratio from above

Ternary case

The main theor

The main theo

The proof

Some notation

Proposition 3

Putting everything

Proposition 3

$$0^{2|w|-|w|_0}3^{2|w|-|w|_3-g(w,0,3)}2^{g(\widetilde{w},1,2)} \quad w \quad 3^{g(w,0,3)}2^{2|w|-|w|_2-g(\widetilde{w},1,2)}1^{2|w|-|w|_1};$$

$$1^{2|w|-|w|_1}2^{2|w|-|w|_2-g(w,1,2)}3^{g(\widetilde{w},0,3)} \quad w \quad 2^{g(w,1,2)}3^{2|w|-|w|_3-g(\widetilde{w},0,3)}0^{2|w|-|w|_0}.$$

The main theore

Definitions
Well-definedness fo

On bounding the MP-ratio from above

Ternary cas

The main theor

Case The main the

The proof

Some notation

Proposition 3

Propositions 4-5

Putting everythin

Proposition 3

$$0^{2|w|-|w|_0} 3^{2|w|-|w|_3-g(w,0,3)} 2^{g(\widetilde{w},1,2)} \ \, w \ \, 3^{g(w,0,3)} 2^{2|w|-|w|_2-g(\widetilde{w},1,2)} 1^{2|w|-|w|_1}; \\ 1^{2|w|-|w|_1} 2^{2|w|-|w|_2-g(w,1,2)} 3^{g(\widetilde{w},0,3)} \ \, w \ \, 2^{g(w,1,2)} 3^{2|w|-|w|_3-g(\widetilde{w},0,3)} 0^{2|w|-|w|_0}.$$

Proposition

At least one among the words $f_1(w)$ and $f_2(w)$ does not contain a subpalindrome of the form 2p2 longer than 2|w|.

On bounding the MP-ratio from above

Ternary case

The main theorem

The main theor

The proof

Some notation

Propositions 4

Putting everythin

Proposition 3

$$0^{2|w|-|w|_0} 3^{2|w|-|w|_3-g(w,0,3)} 2^{g(\widetilde{w},1,2)} \ \, w \ \, 3^{g(w,0,3)} 2^{2|w|-|w|_2-g(\widetilde{w},1,2)} 1^{2|w|-|w|_1}; \\ 1^{2|w|-|w|_1} 2^{2|w|-|w|_2-g(w,1,2)} 3^{g(\widetilde{w},0,3)} \ \, w \ \, 2^{g(w,1,2)} 3^{2|w|-|w|_3-g(\widetilde{w},0,3)} 0^{2|w|-|w|_0}.$$

Proposition

At least one among the words $f_1(w)$ and $f_2(w)$ does not contain a subpalindrome of the form 2p2 longer than 2|w|.

Proof (sketch).

Suppose the contrary: in both the words $f_1(w)$ and $f_2(w)$ a longest subpalindrome of the form 2p2 is longer than 2|w|.

On bounding the MP-ratio from above

Ternary case
The main theore

Quaternary

The main theor

Propositions 1-

Proposition 3
Propositions 4-

Propositions 4–5
Putting everythin

Proposition 3

$$0^{2|w|-|w|_0} 3^{2|w|-|w|_3-g(w,0,3)} 2^{g(\widetilde{w},1,2)} \ \, w \ \, 3^{g(w,0,3)} 2^{2|w|-|w|_2-g(\widetilde{w},1,2)} 1^{2|w|-|w|_1}; \\ 1^{2|w|-|w|_1} 2^{2|w|-|w|_2-g(w,1,2)} 3^{g(\widetilde{w},0,3)} \ \, w \ \, 2^{g(w,1,2)} 3^{2|w|-|w|_3-g(\widetilde{w},0,3)} 0^{2|w|-|w|_0}.$$

Proposition

At least one among the words $f_1(w)$ and $f_2(w)$ does not contain a subpalindrome of the form 2p2 longer than 2|w|.

Proof (sketch).

Suppose the contrary: in both the words $f_1(w)$ and $f_2(w)$ a longest subpalindrome of the form 2p2 is longer than 2|w|. They are of the form $2^{|t|_2+g(\widetilde{w},1,2)}q_w2^{|t|_2+g(\widetilde{w},1,2)}$, where $q_w\in \operatorname{Subw}(w3^{g(w,0,3)})$ and $t\in\operatorname{Pref}(w)$, respectively $2^{|v|_2+g(w,1,2)}p_w2^{|v|_2+g(w,1,2)}$, where $p_w\in\operatorname{Subw}(3^{g(\widetilde{w},0,3)}w)$ and $v\in\operatorname{Suff}(w)$.

subpalindromes
The main theore

Definitions

Well-definedness for n > 4

On bounding the MP-ratio from above

Ternary case
The main theore

Quaternary

The main theor

Propositions 1-

Proposition 3
Propositions 4-

Propositions 4-5

Proposition 3

```
 0^{2|w|-|w|_0} 3^{2|w|-|w|_3-g(w,0,3)} 2^{g(\widetilde{w},1,2)} \  \  \, w \  \, 3^{g(w,0,3)} 2^{2|w|-|w|_2-g(\widetilde{w},1,2)} 1^{2|w|-|w|_1}; \\ 1^{2|w|-|w|_1} 2^{2|w|-|w|_2-g(w,1,2)} 3^{g(\widetilde{w},0,3)} \  \  \, w \  \, 2^{g(w,1,2)} 3^{2|w|-|w|_3-g(\widetilde{w},0,3)} 0^{2|w|-|w|_0}.
```

Proposition

At least one among the words $f_1(w)$ and $f_2(w)$ does not contain a subpalindrome of the form 2p2 longer than 2|w|.

Proof (sketch).

Suppose the contrary: in both the words $f_1(w)$ and $f_2(w)$ a longest subpalindrome of the form 2p2 is longer than 2|w|. They are of the form $2^{|t|_2+g(\widetilde{w},1,2)}q_w2^{|t|_2+g(\widetilde{w},1,2)}$, where $q_w\in \operatorname{Subw}(w3^{g(w,0,3)})$ and $t\in\operatorname{Pref}(w)$, respectively $2^{|v|_2+g(w,1,2)}p_w2^{|v|_2+g(w,1,2)}$, where $p_w\in\operatorname{Subw}(3^{g(\widetilde{w},0,3)}w)$ and $v\in\operatorname{Suff}(w)$.

Putting everythir together

Proposition 3

$$0^{2|w|-|w|_0} 3^{2|w|-|w|_3-g(w,0,3)} 2^{g(\widetilde{w},1,2)} \ \ \, w \ \ \, 3^{g(w,0,3)} 2^{2|w|-|w|_2-g(\widetilde{w},1,2)} 1^{2|w|-|w|_1}; \\ 1^{2|w|-|w|_1} 2^{2|w|-|w|_2-g(w,1,2)} 3^{g(\widetilde{w},0,3)} \ \ \, w \ \ \, 2^{g(w,1,2)} 3^{2|w|-|w|_3-g(\widetilde{w},0,3)} 0^{2|w|-|w|_0}.$$

Proposition

At least one among the words $f_1(w)$ and $f_2(w)$ does not contain a subpalindrome of the form 2p2 longer than 2|w|.

Proof (sketch).

Suppose the contrary: in both the words $f_1(w)$ and $f_2(w)$ a longest subpalindrome of the form 2p2 is longer than 2|w|. They are of the form $2^{|t|_2+g(\widetilde{w},1,2)}q_w2^{|t|_2+g(\widetilde{w},1,2)}$, where $q_w\in \operatorname{Subw}(w3^{g(w,0,3)})$ and $t\in \operatorname{Pref}(w)$, respectively $2^{|v|_2+g(w,1,2)}p_w2^{|v|_2+g(w,1,2)}$, where $p_w\in \operatorname{Subw}(3^{g(\widetilde{w},0,3)}w)$ and $v\in \operatorname{Suff}(w)$.

• $P_{|p_w|} < Q_1$:

Proposition 3

$$0^{2|w|-|w|_0} 3^{2|w|-|w|_3-g(w,0,3)} 2^{g(\widetilde{w},1,2)} \ \ \, w \ \, 3^{g(w,0,3)} 2^{2|w|-|w|_2-g(\widetilde{w},1,2)} 1^{2|w|-|w|_1}; \\ 1^{2|w|-|w|_1} 2^{2|w|-|w|_2-g(w,1,2)} 3^{g(\widetilde{w},0,3)} \ \, w \ \, 2^{g(w,1,2)} 3^{2|w|-|w|_3-g(\widetilde{w},0,3)} 0^{2|w|-|w|_0}.$$

Proposition

At least one among the words $f_1(w)$ and $f_2(w)$ does not contain a subpalindrome of the form 2p2 longer than 2|w|.

Proof (sketch).

Suppose the contrary: in both the words $f_1(w)$ and $f_2(w)$ a longest subpalindrome of the form 2p2 is longer than 2|w|. They are of the form $2^{|t|_2+g(\widetilde{w},1,2)}q_w 2^{|t|_2+g(\widetilde{w},1,2)}$, where $q_w \in \text{Subw}(w3^{g(w,0,3)})$ and $t \in \text{Pref}(w)$, respectively $2^{|v|_2+g(w,1,2)}p_w2^{|v|_2+g(w,1,2)}$, where $p_w \in \text{Subw}(3^{g(\widetilde{w},0,3)}w)$ and $v \in Suff(w)$.

•
$$P_{|p_w|} < Q_1$$
: \checkmark

On bounding the MP-ratio from above

Ternary case

Ouaternary

The main theo

The proof
Propositions 1-

Proposition 3
Propositions 4–8

Propositions 4–5
Putting everythin

Proposition 3

```
 0^{2|w|-|w|_0} 3^{2|w|-|w|_3-g(w,0,3)} 2^{g(\widetilde{w},1,2)} \  \  \, w \  \  \, 3^{g(w,0,3)} 2^{2|w|-|w|_2-g(\widetilde{w},1,2)} 1^{2|w|-|w|_1}; \\ 1^{2|w|-|w|_1} 2^{2|w|-|w|_2-g(w,1,2)} 3^{g(\widetilde{w},0,3)} \  \  \, w \  \  \, 2^{g(w,1,2)} 3^{2|w|-|w|_3-g(\widetilde{w},0,3)} 0^{2|w|-|w|_0}.
```

Proposition

At least one among the words $f_1(w)$ and $f_2(w)$ does not contain a subpalindrome of the form 2p2 longer than 2|w|.

Proof (sketch).

Suppose the contrary: in both the words $f_1(w)$ and $f_2(w)$ a longest subpalindrome of the form 2p2 is longer than 2|w|. They are of the form $2^{|t|_2+g(\widetilde{w},1,2)}q_w2^{|t|_2+g(\widetilde{w},1,2)}$, where $q_w\in \text{Subw}(w3^{g(w,0,3)})$ and $t\in \text{Pref}(w)$, respectively $2^{|v|_2+g(w,1,2)}p_w2^{|v|_2+g(w,1,2)}$, where $p_w\in \text{Subw}(3^{g(\widetilde{w},0,3)}w)$ and $v\in \text{Suff}(w)$.

- $P_{|p_w|} < Q_1$: $\sqrt{}$
- Q is not offset to the right of P:

On bounding the MP-ratio from above

Ternary case

Quaternary

The main theor

Propositions 1–

Proposition 3
Propositions 4–5

Propositions 4–5
Putting everythin

Proposition 3

$$0^{2|w|-|w|_0} 3^{2|w|-|w|_3-g(w,0,3)} 2^{g(\widetilde{w},1,2)} \ \ \, w \ \ \, 3^{g(w,0,3)} 2^{2|w|-|w|_2-g(\widetilde{w},1,2)} 1^{2|w|-|w|_1}; \\ 1^{2|w|-|w|_1} 2^{2|w|-|w|_2-g(w,1,2)} 3^{g(\widetilde{w},0,3)} \ \ \, w \ \ \, 2^{g(w,1,2)} 3^{2|w|-|w|_3-g(\widetilde{w},0,3)} 0^{2|w|-|w|_0}.$$

Proposition

At least one among the words $f_1(w)$ and $f_2(w)$ does not contain a subpalindrome of the form 2p2 longer than 2|w|.

Proof (sketch).

Suppose the contrary: in both the words $f_1(w)$ and $f_2(w)$ a longest subpalindrome of the form 2p2 is longer than 2|w|. They are of the form $2^{|t|_2+g(\overline{w},1,2)}q_w2^{|t|_2+g(\overline{w},1,2)}$, where $q_w\in \operatorname{Subw}(w3^{g(w,0,3)})$ and $t\in\operatorname{Pref}(w)$, respectively $2^{|v|_2+g(w,1,2)}p_w2^{|v|_2+g(w,1,2)}$, where $p_w\in\operatorname{Subw}(3^{g(\overline{w},0,3)}w)$ and $v\in\operatorname{Suff}(w)$.

- $P_{|p_w|} < Q_1$: $\sqrt{}$
- Q is not offset to the right of P: √

Proposition 3

$$0^{2|w|-|w|_0} 3^{2|w|-|w|_3-g(w,0,3)} 2^{g(\widetilde{w},1,2)} \ \ \, w \ \, 3^{g(w,0,3)} 2^{2|w|-|w|_2-g(\widetilde{w},1,2)} 1^{2|w|-|w|_1}; \\ 1^{2|w|-|w|_1} 2^{2|w|-|w|_2-g(w,1,2)} 3^{g(\widetilde{w},0,3)} \ \ \, w \ \, 2^{g(w,1,2)} 3^{2|w|-|w|_3-g(\widetilde{w},0,3)} 0^{2|w|-|w|_0}.$$

Proposition

At least one among the words $f_1(w)$ and $f_2(w)$ does not contain a subpalindrome of the form 2p2 longer than 2|w|.

Proof (sketch).

Suppose the contrary: in both the words $f_1(w)$ and $f_2(w)$ a longest subpalindrome of the form 2p2 is longer than 2|w|. They are of the form $2^{|t|_2+g(\widetilde{w},1,2)}q_w 2^{|t|_2+g(\widetilde{w},1,2)}$, where $q_w \in \text{Subw}(w3^{g(w,0,3)})$ and $t \in \text{Pref}(w)$, respectively $2^{|v|_2+g(w,1,2)} p_w 2^{|v|_2+g(w,1,2)}$, where $p_w \in \text{Subw}(3^{g(\widetilde{w},0,3)}w)$ and $v \in Suff(w)$.

- $P_{|p_{w}|} < Q_1$:
- Q is not offset to the right of P: √
- Q is offset to the right of P:

The main theo

Propositions 1

Proposition 3
Propositions 4–5

Propositions 4–5
Putting everythin

Proposition 3

$$0^{2|w|-|w|_0} 3^{2|w|-|w|_3-g(w,0,3)} 2^{g(\widetilde{w},1,2)} \ \ \, w \ \, 3^{g(w,0,3)} 2^{2|w|-|w|_2-g(\widetilde{w},1,2)} 1^{2|w|-|w|_1}; \\ 1^{2|w|-|w|_1} 2^{2|w|-|w|_2-g(w,1,2)} 3^{g(\widetilde{w},0,3)} \ \ \, w \ \, 2^{g(w,1,2)} 3^{2|w|-|w|_3-g(\widetilde{w},0,3)} 0^{2|w|-|w|_0}.$$

Proposition

At least one among the words $f_1(w)$ and $f_2(w)$ does not contain a subpalindrome of the form 2p2 longer than 2|w|.

Proof (sketch).

Suppose the contrary: in both the words $f_1(w)$ and $f_2(w)$ a longest subpalindrome of the form 2p2 is longer than 2|w|. They are of the form $2^{|t|_2+g(\bar{w},1,2)}q_w2^{|t|_2+g(\bar{w},1,2)}$, where $q_w\in \operatorname{Subw}(w3^{g(w,0,3)})$ and $t\in\operatorname{Pref}(w)$, respectively $2^{|v|_2+g(w,1,2)}p_w2^{|v|_2+g(w,1,2)}$, where $p_w\in\operatorname{Subw}(3^{g(\bar{w},0,3)}w)$ and $v\in\operatorname{Suff}(w)$.

- $P_{|p_w|} < Q_1$: $\sqrt{}$
- Q is not offset to the right of P: √
- Q is offset to the right of P: Time to roll up our sleeves.

Setting the stage

$$3^{g(\widetilde{w},0,3)}w3^{g(w,0,3)}$$

Let $w \in \{0, 1, 2, 3\}^*$, and let m_1, m_2 be nonnegative integers such that $m_1 \leq 3^{g(\widetilde{w},0,3)}$ and $m_2 \leq 3^{g(w,0,3)}$. Let p and q be subpalindromes of $3^{m_1}w$ and $w3^{m_2}$, respectively. Let P and Q be descriptors of p and q (with respect to $3^{m_1}w3^{m_2}$), where $\max P \leq |3^{m_1}w|$ and the smallest m_1 elements of P are $1, 2, \ldots, m_1$, and min $Q \ge m_1 + 1$ and the largest m_2 elements of Q are $|3^{m_1}w| + 1$, $|3^{m_1}w| + 2$, ..., $|3^{m_1}w3^{m_2}|$, and suppose that Q is offset to the right of P.

Definitions
Well-definedness for

On bounding the

MP-ratio from above

Ternary case
The main theorem

Quaternar

The main theore

The proof

Some notation

Proposition 3

Propositions 4–5
Putting everythin

Setting the stage

MP-ratio from above

Proposition 3

Setting the stage

n-ary case

Definitions
Well-definedness for $n \geqslant 4$

On bounding the MP-ratio from above

Ternary case
The main theorem

Quaternary

The main theor

Propositions 1-

Some notation

Proposition 3

Propositions 4–5

Setting the stage

- 1. $j_0 = m_1 + 1$;
- 2. assuming that j_i is defined, let $\overrightarrow{j_i}$ be the smallest integer, if it exists, such that $\overrightarrow{j_i} \geqslant j_i$ and $\overrightarrow{j_i} \in P$, and if $\sigma_P(\overrightarrow{j_i}) > m_1$, then let $j_i' = \sigma_P(\overrightarrow{j_i})$; otherwise (that is, if $\overrightarrow{j_i}$ does not exist, or if $\sigma_P(\overrightarrow{j_i}) \leqslant m_1$), j_i' is undefined and the procedure stops;

The main theor

Definitions
Well-definedness for n ≥ 4

On bounding the MP-ratio from above

Ternary case
The main theore

case

The proof

Some notation

Proposition 3

Putting everythin

Setting the stage

- 1. $j_0 = m_1 + 1$;
- 2. assuming that j_i is defined, let $\overrightarrow{\jmath_i}$ be the smallest integer, if it exists, such that $\overrightarrow{\jmath_i} \geqslant j_i$ and $\overrightarrow{\jmath_i} \in P$, and if $\sigma_P(\overrightarrow{\jmath_i}) > m_1$, then let $j_i' = \sigma_P(\overrightarrow{\jmath_i})$; otherwise (that is, if $\overrightarrow{\jmath_i}$ does not exist, or if $\sigma_P(\overrightarrow{\jmath_i}) \leqslant m_1$), j_i' is undefined and the procedure stops;
- 3. assuming that j_i' is defined, let $\overleftarrow{j_i'}$ be the largest integer, if it exists, such that $\overleftarrow{j_i'} \leqslant j_i'$ and $\overleftarrow{j_i'} \in Q$, and if $\sigma_Q(\overleftarrow{j_i'}) \leqslant |3^{m_1}w|$, then let $j_{i+1} = \sigma_Q(\overleftarrow{j_i'})$; otherwise (that is, if $\overleftarrow{j_i'}$ does not exist, or if $\sigma_Q(\overleftarrow{j_i'}) > |3^{m_1}w|$), j_{i+1} is undefined and the procedure stops.

Setting the stage

- 1. $j_0 = m_1 + 1$;
- 2. assuming that j_i is defined, let $\overrightarrow{j_i}$ be the smallest integer, if it exists, such that $\overrightarrow{j_i} \ge i_i$ and $\overrightarrow{j_i} \in P$, and if $\sigma_P(\overrightarrow{j_i}) > m_1$, then let $i'_i = \sigma_P(\overrightarrow{j_i})$; otherwise (that is, if $\overrightarrow{j_i}$ does not exist, or if $\sigma_P(\overrightarrow{j_i}) \leqslant m_1$), j_i' is undefined and the procedure stops:
- 3. assuming that j'_i is defined, let $\overleftarrow{j'_i}$ be the largest integer, if it exists, such that $\overleftarrow{\jmath_{i}'}\leqslant j_{i}'$ and $\overleftarrow{\jmath_{i}'}\in \mathcal{Q}$, and if $\sigma_{\mathcal{Q}}(\overleftarrow{\jmath_{i}'})\leqslant |3^{m_{1}}w|$, then let $j_{i+1} = \sigma_Q(\overleftarrow{j_i})$; otherwise (that is, if $\overleftarrow{j_i}$ does not exist, or if $\sigma_O(\frac{1}{i}) > |3^{m_1}w|$), j_{i+1} is undefined and the procedure stops.

Setting the stage

- 1. $j_0 = m_1 + 1$;
- 2. assuming that j_i is defined, let $\overrightarrow{j_i}$ be the smallest integer, if it exists, such that $\overrightarrow{j_i} \ge i_i$ and $\overrightarrow{j_i} \in P$, and if $\sigma_P(\overrightarrow{j_i}) > m_1$, then let $i'_i = \sigma_P(\overrightarrow{j_i})$; otherwise (that is, if $\overrightarrow{j_i}$ does not exist, or if $\sigma_P(\overrightarrow{j_i}) \leqslant m_1$), j_i' is undefined and the procedure stops:
- 3. assuming that j'_i is defined, let $\overleftarrow{j'_i}$ be the largest integer, if it exists, such that $\overleftarrow{\jmath_{i}'}\leqslant j_i'$ and $\overleftarrow{\jmath_{i}'}\in \mathcal{Q}$, and if $\sigma_{\mathcal{Q}}(\overleftarrow{\jmath_{i}'})\leqslant |3^{m_1}w|$, then let $j_{i+1} = \sigma_Q(\overleftarrow{j_i})$; otherwise (that is, if $\overleftarrow{j_i}$ does not exist, or if $\sigma_O(\frac{1}{i}) > |3^{m_1}w|$), j_{i+1} is undefined and the procedure stops.

Setting the stage

- 1. $j_0 = m_1 + 1$;
- 2. assuming that j_i is defined, let $\overrightarrow{j_i}$ be the smallest integer, if it exists, such that $\overrightarrow{j_i} \ge i_i$ and $\overrightarrow{j_i} \in P$, and if $\sigma_P(\overrightarrow{j_i}) > m_1$, then let $i'_i = \sigma_P(\overrightarrow{j_i})$; otherwise (that is, if $\overrightarrow{j_i}$ does not exist, or if $\sigma_P(\overrightarrow{j_i}) \leqslant m_1$), j_i' is undefined and the procedure stops:
- 3. assuming that j'_i is defined, let $\overleftarrow{j'_i}$ be the largest integer, if it exists, such that $\overleftarrow{\jmath_{i}'}\leqslant j_{i}'$ and $\overleftarrow{\jmath_{i}'}\in \mathcal{Q}$, and if $\sigma_{\mathcal{Q}}(\overleftarrow{\jmath_{i}'})\leqslant |3^{m_{1}}w|$, then let $j_{i+1} = \sigma_Q(\overleftarrow{j_i})$; otherwise (that is, if $\overleftarrow{j_i}$ does not exist, or if $\sigma_O(\frac{1}{i}) > |3^{m_1}w|$), j_{i+1} is undefined and the procedure stops.

Setting the stage

- 1. $j_0 = m_1 + 1$;
- 2. assuming that j_i is defined, let $\overrightarrow{j_i}$ be the smallest integer, if it exists, such that $\overrightarrow{j_i} \ge i_i$ and $\overrightarrow{j_i} \in P$, and if $\sigma_P(\overrightarrow{j_i}) > m_1$, then let $i'_i = \sigma_P(\overrightarrow{j_i})$; otherwise (that is, if $\overrightarrow{j_i}$ does not exist, or if $\sigma_P(\overrightarrow{j_i}) \leqslant m_1$), j_i' is undefined and the procedure stops:
- 3. assuming that j'_i is defined, let $\overleftarrow{j'_i}$ be the largest integer, if it exists, such that $\overleftarrow{\jmath_{i}'}\leqslant j_i'$ and $\overleftarrow{\jmath_{i}'}\in \mathcal{Q}$, and if $\sigma_{\mathcal{Q}}(\overleftarrow{\jmath_{i}'})\leqslant |3^{m_1}w|$, then let $j_{i+1} = \sigma_Q(\overleftarrow{j_i})$; otherwise (that is, if $\overleftarrow{j_i}$ does not exist, or if $\sigma_O(\frac{1}{i}) > |3^{m_1}w|$), j_{i+1} is undefined and the procedure stops.

The main theore

Definitions
Well-definedness for $n \ge 4$

On bounding the MP-ratio from abov

Ternary case
The main theore

Case
The main theore

The proof

Some notation
Proposition 3

Proposition 3
Propositions 4

Putting everythin together

Setting the stage

- 1. $j_0 = m_1 + 1$;
- 2. assuming that j_i is defined, let $\overrightarrow{j_i}$ be the smallest integer, if it exists, such that $\overrightarrow{j_i} \geqslant j_i$ and $\overrightarrow{j_i} \in P$, and if $\sigma_P(\overrightarrow{j_i}) > m_1$, then let $j_i' = \sigma_P(\overrightarrow{j_i})$; otherwise (that is, if $\overrightarrow{j_i}$ does not exist, or if $\sigma_P(\overrightarrow{j_i}) \leqslant m_1$), j_i' is undefined and the procedure stops;
- 3. assuming that j_i' is defined, let $\overleftarrow{j_i'}$ be the largest integer, if it exists, such that $\overleftarrow{j_i'} \leqslant j_i'$ and $\overleftarrow{j_i'} \in Q$, and if $\sigma_Q(\overleftarrow{j_i'}) \leqslant |3^{m_1}w|$, then let $j_{i+1} = \sigma_Q(\overleftarrow{j_i'})$; otherwise (that is, if $\overleftarrow{j_i'}$ does not exist, or if $\sigma_Q(\overleftarrow{j_i'}) > |3^{m_1}w|$), j_{i+1} is undefined and the procedure stops.

The main the

Definitions
Well-definedness for $n \ge 4$

Ternary case

Ouaternary

The main theorem

Propositions 1–

Proposition 3
Propositions 4–5

Propositions 4–5
Putting everythin

Setting the stage

- 1. $j_0 = m_1 + 1$;
- 2. assuming that j_i is defined, let $\overrightarrow{j_i}$ be the smallest integer, if it exists, such that $\overrightarrow{j_i} \geqslant j_i$ and $\overrightarrow{j_i} \in P$, and if $\sigma_P(\overrightarrow{j_i}) > m_1$, then let $j_i' = \sigma_P(\overrightarrow{j_i})$; otherwise (that is, if $\overrightarrow{j_i}$ does not exist, or if $\sigma_P(\overrightarrow{j_i}) \leqslant m_1$), j_i' is undefined and the procedure stops;
- 3. assuming that j_i' is defined, let $\overleftarrow{j_i'}$ be the largest integer, if it exists, such that $\overleftarrow{j_i'} \leqslant j_i'$ and $\overleftarrow{j_i'} \in Q$, and if $\sigma_Q(\overleftarrow{j_i'}) \leqslant |3^{m_1}w|$, then let $j_{i+1} = \sigma_Q(\overleftarrow{j_i'})$; otherwise (that is, if $\overleftarrow{j_i'}$ does not exist, or if $\sigma_Q(\overleftarrow{j_i'}) > |3^{m_1}w|$), j_{i+1} is undefined and the procedure stops.

On bounding the MP-ratio from above

The main theore

Case
The main theore

Propositions 1–3

Proposition 3
Propositions 4-5

Propositions 4–5
Putting everythin

Setting the stage

- 1. $j_0 = m_1 + 1$;
- 2. assuming that j_i is defined, let $\overrightarrow{j_i}$ be the smallest integer, if it exists, such that $\overrightarrow{j_i} \geqslant j_i$ and $\overrightarrow{j_i} \in P$, and if $\sigma_P(\overrightarrow{j_i}) > m_1$, then let $j_i' = \sigma_P(\overrightarrow{j_i})$; otherwise (that is, if $\overrightarrow{j_i}$ does not exist, or if $\sigma_P(\overrightarrow{j_i}) \leqslant m_1$), j_i' is undefined and the procedure stops;
- 3. assuming that j_i' is defined, let $\overleftarrow{j_i'}$ be the largest integer, if it exists, such that $\overleftarrow{j_i'} \leqslant j_i'$ and $\overleftarrow{j_i'} \in Q$, and if $\sigma_Q(\overleftarrow{j_i'}) \leqslant |3^{m_1}w|$, then let $j_{i+1} = \sigma_Q(\overleftarrow{j_i'})$; otherwise (that is, if $\overleftarrow{j_i'}$ does not exist, or if $\sigma_Q(\overleftarrow{j_i'}) > |3^{m_1}w|$), j_{i+1} is undefined and the procedure stops.

The main the

Definitions

Well-definedness for n ≥ 4

On bounding the

Ternary case

The main theore

The main theorem

Propositions 1–3

Proposition 3 Propositions 4–

Propositions 4–5
Putting everythin together

Setting the stage

- 1. $j_0 = m_1 + 1$;
- 2. assuming that j_i is defined, let $\overrightarrow{\jmath_i}$ be the smallest integer, if it exists, such that $\overrightarrow{\jmath_i} \geqslant j_i$ and $\overrightarrow{\jmath_i} \in P$, and if $\sigma_P(\overrightarrow{\jmath_i}) > m_1$, then let $j_i' = \sigma_P(\overrightarrow{\jmath_i})$; otherwise (that is, if $\overrightarrow{\jmath_i}$ does not exist, or if $\sigma_P(\overrightarrow{\jmath_i}) \leqslant m_1$), j_i' is undefined and the procedure stops;
- 3. assuming that j_i' is defined, let $\overleftarrow{j_i'}$ be the largest integer, if it exists, such that $\overleftarrow{j_i'} \leqslant j_i'$ and $\overleftarrow{j_i'} \in Q$, and if $\sigma_Q(\overleftarrow{j_i'}) \leqslant |3^{m_1}w|$, then let $j_{i+1} = \sigma_Q(\overleftarrow{j_i'})$; otherwise (that is, if $\overleftarrow{j_i'}$ does not exist, or if $\sigma_Q(\overleftarrow{j_i'}) > |3^{m_1}w|$), j_{i+1} is undefined and the procedure stops.

The main

Definitions
Well-definedness for n ≥ 4
On bounding the

Ternary case

Quaternary

The main theorer

Propositions 1– Some notation Proposition 3

Propositions 4–

Putting everythin together

Setting the stage

- 1. $j_0 = m_1 + 1$;
- 2. assuming that j_i is defined, let $\overrightarrow{j_i}$ be the smallest integer, if it exists, such that $\overrightarrow{j_i} \geqslant j_i$ and $\overrightarrow{j_i} \in P$, and if $\sigma_P(\overrightarrow{j_i}) > m_1$, then let $j_i' = \sigma_P(\overrightarrow{j_i})$; otherwise (that is, if $\overrightarrow{j_i}$ does not exist, or if $\sigma_P(\overrightarrow{j_i}) \leqslant m_1$), j_i' is undefined and the procedure stops;
- 3. assuming that j_i' is defined, let $\overleftarrow{j_i'}$ be the largest integer, if it exists, such that $\overleftarrow{j_i'} \leqslant j_i'$ and $\overleftarrow{j_i'} \in Q$, and if $\sigma_Q(\overleftarrow{j_i'}) \leqslant |3^{m_1}w|$, then let $j_{i+1} = \sigma_Q(\overleftarrow{j_i'})$; otherwise (that is, if $\overleftarrow{j_i'}$ does not exist, or if $\sigma_Q(\overleftarrow{j_i'}) > |3^{m_1}w|$), j_{i+1} is undefined and the procedure stops.

Kristina Ago

Introduction

Rinary cae

Definitions

Some interesting roles of

subpalindromes

II-ai y Cai

Well-definedness

n ≥ 4

On bounding the MP-ratio from above

Ternary case

THE HIGH HIGO

Caca

The main theore

Propositions 1=

Some notation

Proposition 3

Putting everythin

Dissection

Dissection

Definition

Let τ be the largest integer such that j_{τ} (from the previous procedure) is defined. Then the sequence $x_1, x_2, \dots, x_{\tau+1}$, with $x_i = z[j_{i-1}, j_i - 1]$ for $1 \le i \le \tau$ and $x_{\tau+1} = z[j_{\tau}, |3^{m_1}w|]$, is called the dissection of w, and the value au is called the tally of the dissection (clearly, those notions depend not only on w but also on m_1 , m_2 , P, Q, but all these parameters will always be clear from the context).

Definition

Let τ be the largest integer such that j_{τ} (from the previous procedure) is defined. Then the sequence $x_1, x_2, \dots, x_{\tau+1}$, with $x_i = z[j_{i-1}, j_i - 1]$ for $1 \le i \le \tau$ and $x_{\tau+1} = z[j_{\tau}, |3^{m_1}w|]$, is called the dissection of w, and the value au is called the tally of the dissection (clearly, those notions depend not only on w but also on m_1 , m_2 , P, Q, but all these parameters will always be clear from the context).

Definition

Let τ be the largest integer such that j_{τ} (from the previous procedure) is defined. Then the sequence $x_1, x_2, \dots, x_{\tau+1}$, with $x_i = z[j_{i-1}, j_i - 1]$ for $1 \le i \le \tau$ and $x_{\tau+1} = z[j_{\tau}, |3^{m_1}w|]$, is called the dissection of w, and the value au is called the tally of the dissection (clearly, those notions depend not only on w but also on m_1 , m_2 , P, Q, but all these parameters will always be clear from the context).

$$\underbrace{3}_{1} \underbrace{3}_{1} \underbrace{3}_{1} \underbrace{3}_{1} \underbrace{0}_{1} \underbrace{0}_{1}$$

Kristina Ago

roles of

 $n \ge 4$

MP-ratio from above

Proposition 3

Refined dissection

Definition

Refined dissection

Definition

We also define the refined dissection of w as follows. Let $j_0^{\diamond}, j_1^{\diamond}, j_2^{\diamond}, \dots$ be the sequence defined in the same way as j_0, j_1, j_2, \dots but with $j_0^{\diamond} = \sigma_O(|\overline{3^{m_1}w}|)$ instead of $j_0 = m_1 + 1$. Let $x'_i = z[j_{i-1}, j_{i-1}^{\diamond} - 1]$ and $x''_i = z[j_{i-1}^{\diamond}, j_i - 1]$ for $1 \leq i \leq \tau$, while $x'_{\tau+1} = z[j_{\tau}, j_{\tau}^{\diamond} - 1]$ and $x''_{\tau+1} = z[j_{\tau}^{\diamond}, |3^{m_1}w|]$ if j_{τ}^{\diamond} is defined, respectively $x'_{\tau+1} = z[j_{\tau}, |3^{m_1}w|]$ if j_{τ}^{\diamond} is undefined (and $x_{\tau+1}^{"}$ is undefined in this last case). Then the sequence $x'_1, x''_1, x'_2, x''_2, \dots$ is the refined dissection of w.

Refined dissection

Definition

We also define the refined dissection of w as follows. Let $j_0^{\diamond}, j_1^{\diamond}, j_2^{\diamond}, \dots$ be the sequence defined in the same way as j_0, j_1, j_2, \dots but with $j_0^{\diamond} = \sigma_O(|3^{m_1}w|)$ instead of $j_0 = m_1 + 1$. Let $x'_i = z[j_{i-1}, j_{i-1}^{\diamond} - 1]$ and $x''_i = z[j_{i-1}^{\diamond}, j_i - 1]$ for $1 \leq i \leq \tau$, while $x'_{\tau+1} = z[j_{\tau}, j_{\tau}^{\diamond} - 1]$ and $x''_{\tau+1} = z[j_{\tau}^{\diamond}, |3^{m_1}w|]$ if j_{τ}^{\diamond} is defined, respectively $x'_{\tau+1} = z[j_{\tau}, |3^{m_1}w|]$ if j_{τ}^{\diamond} is undefined (and $x''_{\tau+1}$ is undefined in this last case). Then the sequence $x'_1, x''_1, x''_2, x''_2, \dots$ is the refined dissection of w.

Refined dissection

Definition

We also define the refined dissection of w as follows. Let $j_0^{\diamond}, j_1^{\diamond}, j_2^{\diamond}, \dots$ be the sequence defined in the same way as j_0, j_1, j_2, \dots but with $j_0^{\diamond} = \sigma_O(|3^{m_1}w|)$ instead of $j_0 = m_1 + 1$. Let $x'_i = z[j_{i-1}, j_{i-1}^{\diamond} - 1]$ and $x''_i = z[j_{i-1}^{\diamond}, j_i - 1]$ for $1 \leq i \leq \tau$, while $x'_{\tau+1} = z[j_{\tau}, j_{\tau}^{\diamond} - 1]$ and $x''_{\tau+1} = z[j_{\tau}^{\diamond}, |3^{m_1}w|]$ if j_{τ}^{\diamond} is defined, respectively $x'_{\tau+1} = z[j_{\tau}, |3^{m_1}w|]$ if j_{τ}^{\diamond} is undefined (and $x_{\tau+1}^{"}$ is undefined in this last case). Then the sequence $x'_1, x''_1, x''_2, x''_2, \dots$ is the refined dissection of w.

Refined dissection

Definition

We also define the refined dissection of w as follows. Let $j_0^{\diamond}, j_1^{\diamond}, j_2^{\diamond}, \dots$ be the sequence defined in the same way as j_0, j_1, j_2, \dots but with $j_0^{\diamond} = \sigma_O(|\overline{3^{m_1}w}|)$ instead of $j_0 = m_1 + 1$. Let $x'_i = z[j_{i-1}, j_{i-1}^{\diamond} - 1]$ and $x''_i = z[j_{i-1}^{\diamond}, j_i - 1]$ for $1 \leq i \leq \tau$, while $x'_{\tau+1} = z[j_{\tau}, j_{\tau}^{\diamond} - 1]$ and $x''_{\tau+1} = z[j_{\tau}^{\diamond}, |3^{m_1}w|]$ if j_{τ}^{\diamond} is defined, respectively $x'_{\tau+1} = z[j_{\tau}, |3^{m_1}w|]$ if j_{τ}^{\diamond} is undefined (and $x''_{\tau+1}$ is undefined in this last case). Then the sequence $x'_1, x''_1, x'_2, x''_2, \dots$ is the refined dissection of w.

Kristina Ago

Introduction

Definitions

Some interes

roles of

The main theore

n-ary cas

Definitions

n ≥ 4

On bounding the MP-ratio from above

Ternary case

The main theor

Quatornary

The main these

The proof

Propositions 1

Some notation

Proposition 3

Putting everythin

Further preparation

Some interest

roles of

The main theore

n-ary cas

Definitions

Well-definedness for $n \ge 4$

On bounding the

MP-ratio from above

Ternary case
The main theore

Quatornari

The main theor

The proof

Some notation

Proposition 3

Propositions 4-

Putting everything together

Further preparation

• Let $z = 3^{m_1} w 3^{m_2}$, $w \in \{0, 3\}^*$.

MP-ratio from above

Proposition 3

Further preparation

- Let $z = 3^{m_1} w 3^{m_2}$, $w \in \{0, 3\}^*$.
- On the set $\{1, 2, \ldots, |z|\}$, define:

 $s \sim s'$ if there exists $i \in \mathbb{Z}$ such that $s' = (\sigma_Q \circ \sigma_P)^i(s)$.

Ternary cas

The main theo

Case
The main theory

The proof

Some notation

Proposition 3
Propositions 4

Putting everythin

Further preparation

- Let $z = 3^{m_1} w 3^{m_2}$, $w \in \{0, 3\}^*$.
- On the set {1,2,...,|z|}, define:

$$s \sim s'$$
 if there exists $i \in \mathbb{Z}$ such that $s' = (\sigma_Q \circ \sigma_P)^i(s)$.

If s ∈ P/Q, then the equivalence classes [s]~ and [σ_{P/Q}(s)]~ will be called twins.

Case

The proof

Some notation

Proposition 3

Putting everything

Further preparation

- Let $z = 3^{m_1} w 3^{m_2}$, $w \in \{0, 3\}^*$.
- On the set {1,2,...,|z|}, define:

$$s \sim s'$$
 if there exists $i \in \mathbb{Z}$ such that $s' = (\sigma_Q \circ \sigma_P)^i(s)$.

- If s ∈ P/Q, then the equivalence classes [s]~ and [σ_{P/Q}(s)]~ will be called twins.
- A class that is not the twin of any other class will be called twinless.

Ternary case

The main theore

case

The proof

Some notation

Proposition 3
Propositions 4-

Putting everythin

Further preparation

- Let $z = 3^{m_1} w 3^{m_2}$, $w \in \{0, 3\}^*$.
- On the set {1,2,...,|z|}, define:

$$s \sim s'$$
 if there exists $i \in \mathbb{Z}$ such that $s' = (\sigma_Q \circ \sigma_P)^i(s)$.

- If s ∈ P/Q, then the equivalence classes [s]~ and [σ_{P/Q}(s)]~ will be called twins.
- A class that is not the twin of any other class will be called twinless.
- We also define the following relation:

 $s \frown s'$ if s and s' are in the same \sim -class or in two twin classes.

Putting everythin together

Further preparation

- Let $z = 3^{m_1} w 3^{m_2}$, $w \in \{0, 3\}^*$.
- On the set {1, 2, ..., |z|}, define:

$$s \sim s'$$
 if there exists $i \in \mathbb{Z}$ such that $s' = (\sigma_Q \circ \sigma_P)^i(s)$.

- If s ∈ P/Q, then the equivalence classes [s]~ and [σ_{P/Q}(s)]~ will be called twins.
- A class that is not the twin of any other class will be called twinless.
- We also define the following relation:

 $s \frown s'$ if s and s' are in the same \sim -class or in two twin classes.

• Let $M_1 = [1, m_1]_{\mathbb{N}}$, $M_2 = [|w| + m_1 + 1, |z|]_{\mathbb{N}}$, $M = M_1 \cup M_2$, and let

$$M^{\otimes} = \{ s \in M : [s] \setminus M \subseteq P \cap Q \},$$

with M_1^{\oplus} and M_2^{\oplus} similarly defined.

Introduction

Binary car

Definitions
Some interesting roles of

subpalindro

The main the

n-ary cas

Well-definedness t

On bounding the

MP-ratio from above

Ternary cas

The main theor

Quaterr

The main theore

The proof

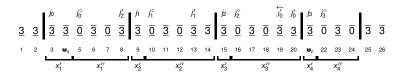
Some notation

Proposition 3

Propositions 4–5

Putting everything

Further preparation



Ternary cas

The main theor

Quaterna

The main theore

The proof

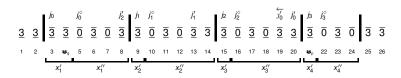
Propositions 1 Some notation

Proposition 3

Propositions 4-

Putting everythin

Further preparation



The mappings σ_P and σ_Q act as follows:

$$1\overset{\sigma_{P}}{\mapsto}23\overset{\sigma_{Q}}{\mapsto}6\overset{\sigma_{P}}{\mapsto}17\overset{\sigma_{Q}}{\mapsto}11\overset{\sigma_{P}}{\mapsto}11\overset{\sigma_{Q}}{\mapsto}17\overset{\sigma_{P}}{\mapsto}6\overset{\sigma_{Q}}{\mapsto}23\overset{\sigma_{P}}{\mapsto}1;$$

$$2\stackrel{\sigma_{P}}{\mapsto}21\stackrel{\sigma_{Q}}{\mapsto}8\stackrel{\sigma_{P}}{\mapsto}15\stackrel{\sigma_{Q}}{\mapsto}13\stackrel{\sigma_{P}}{\mapsto}9\stackrel{\sigma_{Q}}{\mapsto}19\stackrel{\sigma_{P}}{\mapsto}4\stackrel{\sigma_{Q}}{\mapsto}25;$$

$$26 \stackrel{\sigma_Q}{\mapsto} 3 \stackrel{\sigma_P}{\mapsto} 20;$$

$$14 \stackrel{\sigma_Q}{\mapsto} 14;$$

$$24 \overset{\sigma_{\mathcal{O}}}{\mapsto} 5 \overset{\sigma_{\mathcal{P}}}{\mapsto} 18 \overset{\sigma_{\mathcal{O}}}{\mapsto} 10 \overset{\sigma_{\mathcal{P}}}{\mapsto} 12 \overset{\sigma_{\mathcal{O}}}{\mapsto} 16 \overset{\sigma_{\mathcal{P}}}{\mapsto} 7 \overset{\sigma_{\mathcal{O}}}{\mapsto} 22.$$

Proposition 3

Further preparation

$$\underbrace{3}_{1} \underbrace{3}_{2} \underbrace{3}_{3} \underbrace{6}_{3} \underbrace{7}_{10} \underbrace{7}_{1$$

The mappings σ_P and σ_Q act as follows:

$$1\overset{\sigma_{P}}{\mapsto}23\overset{\sigma_{Q}}{\mapsto}6\overset{\sigma_{P}}{\mapsto}17\overset{\sigma_{Q}}{\mapsto}11\overset{\sigma_{P}}{\mapsto}11\overset{\sigma_{Q}}{\mapsto}17\overset{\sigma_{P}}{\mapsto}6\overset{\sigma_{Q}}{\mapsto}23\overset{\sigma_{P}}{\mapsto}1;$$

$$2\stackrel{\sigma_{P}}{\mapsto}21\stackrel{\sigma_{Q}}{\mapsto}8\stackrel{\sigma_{P}}{\mapsto}15\stackrel{\sigma_{Q}}{\mapsto}13\stackrel{\sigma_{P}}{\mapsto}9\stackrel{\sigma_{Q}}{\mapsto}19\stackrel{\sigma_{P}}{\mapsto}4\stackrel{\sigma_{Q}}{\mapsto}25;$$

$$26 \stackrel{\sigma_Q}{\mapsto} 3 \stackrel{\sigma_P}{\mapsto} 20;$$

$$14 \stackrel{\sigma_Q}{\mapsto} 14;$$

$$24 \overset{\sigma_Q}{\mapsto} 5 \overset{\sigma_P}{\mapsto} 18 \overset{\sigma_Q}{\mapsto} 10 \overset{\sigma_P}{\mapsto} 12 \overset{\sigma_Q}{\mapsto} 16 \overset{\sigma_P}{\mapsto} 7 \overset{\sigma_Q}{\mapsto} 22.$$

$$M^{\textcircled{5}} = \{1, 2, 25\}.$$

The main theor

Propositions 1

Some notation

Proposition 3
Propositions 4

Putting everythi

Little gaps

For $a \in \{0,3\}$, we introduce the following notation:

$$\Xi_a = \bigcup_{X \in \{P,Q\}} (\{m_1 + 1 \leqslant s \leqslant |3^{m_1}w| : z[s] = a\} \setminus X).$$

For $a \in \{0, 3\}$, we introduce the following notation:

$$\Xi_a = \bigcup_{X \in \{P,Q\}} (\{m_1 + 1 \leqslant s \leqslant |3^{m_1}w| : z[s] = a\} \setminus X).$$

$$\Xi_0 = \{22,24\} \text{ and } \Xi_3 = \{14,20\}.$$

Proposition 3

Little gaps

We have the following bounds on $|\Xi_a|$:

- a) $|\Xi_0|$ is greater than or equal to the total number of $0-\sim$ -classes.
- b) $|\Xi_3|$ is greater than or equal to the sum of the following two values:
 - the number of 3-\(\sigma\)-classes that contain some element of $M \setminus M^{\Theta}$:
 - the number of 3-\(\sigma\)-classes that do not contain any element of M.

Introduction

Definitions
Some interesting roles of subpalindromes

Definitions
Well-definedness for $n \ge 4$

On bounding the MP-ratio from abov

Ternary case
The main theorem

Quaternar

The main theor The proof

Propositions 1-Some notation

Proposition 3
Propositions 4–5

Putting everythin

We have the following bounds on $|\Xi_a|$:

- a) $|\Xi_0|$ is greater than or equal to the total number of $0-\sim$ -classes.
- b) $|\Xi_3|$ is greater than or equal to the sum of the following two values:
 - the number of 3 ~-classes that contain some element of M \ M[®];
 - the number of 3 —-classes that do not contain any element of M.

In fact, we have:

$$|\Xi_0| + |\Xi_3| \geqslant \frac{|w|_3}{2(\tau+1)}.$$

Proposition 3

Lemma

Let $w \in \{0,3\}^*$, and let m_1, m_2 be nonnegative integers such that $m_1 \leqslant 3^{g(\widetilde{w},0,3)}$ and $m_2 \leqslant 3^{g(w,0,3)}$. Let p and q be subpalindromes of 3^{m_1} w and $w3^{m_2}$, respectively. Let P and Q be descriptors of p and q (with respect to 3^{m_1} w 3^{m_2}). where max $P \leq |3^{m_1}w|$ and the smallest m_1 elements of P are 1, 2, ..., m_1 , and min $Q \ge m_1 + 1$ and the largest m_2 elements of Q are $|3^{m_1}w| + 1$, $|3^{m_1}w| + 2$, ..., $|3^{m_1}w3^{m_2}|$, and suppose that Q is offset to the right of P. Then

$$|p|+|q| \leqslant 2|w|+m-\frac{|w|_3}{2(\tau+1)},$$

where $m = m_1 + m_2$ and τ is the tally of the dissection of w.

Proposition 3

Lemma

Let $w \in \{0, 1, 2, 3\}^*$ where $|w|_0 \le |w|_1 \le |w|_2 \le |w|_3$, and let m_1 , m_2 be nonnegative integers such that $m_1 \leq 3^{g(w,0,3)}$ and $m_2 \leq 3^{g(w,0,3)}$. Let p and q be subpalindromes of 3^{m_1} w and $w3^{m_2}$, respectively. Let P and Q be descriptors of p and g (with respect to $3^{m_1}w3^{m_2}$), where max $P \leq |3^{m_1}w|$ and the smallest m_1 elements of P are $1, 2, ..., m_1$, and min $Q \ge m_1 + 1$ and the largest m_2 elements of Q are $|3^{m_1}w|+1$, $|3^{m_1}w|+2$, ..., $|3^{m_1}w3^{m_2}|$. Suppose that Q is offset to the right of P. Then, if $u = (3^{m_1}w3^{m_2})[Q_1, P_{|D|}]$ if $Q_1 \leqslant P_{|p|}$, and $u = \varepsilon$ otherwise, we have

$$|p|+|q| \leqslant 4|w|_3+2|u|_2+4|w|_0.$$

Introduction

Definitions

0----

roles of

The main the see

n_arv cas

II-aly Cas

Definitions

n ≥ 4

On bounding the MP-ratio from above

Ternary cas

THE HIGH HIGO

case

The proof

Propositions 1-

Some notation

Propositions 4-5

Putting everythin

Propositions 4–5

Introduction

Rinary cas

Definitions
Some interest

Some interesting roles of

subpalindrome

The main theor

Definitions

n ≥ 4

On bounding the MP-ratio from abo

Ternary case
The main theore

Quaternary

The main theo

The proof

Some notation

Propositions 4-5

Putting everythin

Propositions 4–5

Proposition

At least one among the words $f_1(w)$ and $f_2(w)$ does not contain a subpalindrome of the form 3p3 longer than 2|w|.

Propositions 4-5

Propositions 4–5

Proposition

At least one among the words $f_1(w)$ and $f_2(w)$ does not contain a subpalindrome of the form 3p3 longer than 2|w|.

Proposition

One of the following is true: the word $f_1(w)$ does not contain a subpalindrome of the form 2p2 longer than 2|w|, or the word $f_2(w)$ does not contain a subpalindrome of the form 3p3 longer than 2|w|.

Propositions 4-5

Propositions 4–5

Proposition

At least one among the words $f_1(w)$ and $f_2(w)$ does not contain a subpalindrome of the form 3p3 longer than 2|w|.

Proposition

One of the following is true: the word $f_1(w)$ does not contain a subpalindrome of the form 2p2 longer than 2|w|, or the word $f_2(w)$ does not contain a subpalindrome of the form 3p3 longer than 2|w|. Analogously, the word $f_1(w)$ does not contain a subpalindrome of the form 3p3 longer than 2|w|, or the word $f_2(w)$ does not contain a subpalindrome of the form 2p2 longer than 2|w|.

Introduction

Rinary cae

Definitions

Some interesting

subpalindromes

rrai y ca

Well-definedness t

n ≥ 4 On bounding the

MP-ratio from above

Ternary case

The main the an

The proof

Propositions 1-

Some notation

Putting everything together

Putting everything together

Introduction

Binary cas

Definitions
Some interestir

The main theore

1110 1110111 111001

Definitions Well-definedness for

On bounding the

MP-ratio from above

Ternary case
The main theorem

Quaternary

The main theore

The proof

Some notation Proposition 3

Propositions 4-

Putting everything together

Putting everything together

Introduction

Definitions
Some interesti

Some interesting roles of subpalindromes

The main theor

n-ary case
Definitions
Well-definedness f

On bounding the MP-ratio from abov

Ternary case

The main theore

case

The proof

Some notation

Propositions 4

Putting everything together

Putting everything together

	$f_1(w)$	$f_2(w)$	
0 <i>p</i> 0			
1 <i>p</i> 1			
2 <i>p</i> 2			
3 <i>p</i> 3			

Introduction

Definitions
Some interesti

Some interestin

The main theor

Definitions
Well-definedness for

On bounding the MP-ratio from abov

Ternary case

The main theore

The main theor

The proof

Some notation

Propositions 4

Propositions 4–5
Putting everything together

Putting everything together

	$f_1(w)$	$f_2(w)$
0 <i>p</i> 0	×	×
1 <i>p</i> 1		
2 <i>p</i> 2		
3 <i>p</i> 3		

Introduction

Definitions
Some interesti

Some interesting roles of subpalindromes

The main theo

Definitions
Well-definedness

On bounding the

Ternary case

The main theore

Quaternary

The main theo

The proof

Some notation

Bronseitions 4

Putting everything together

Putting everything together

	$f_1(w)$	$f_2(w)$
0 <i>p</i> 0	×	×
1 <i>p</i> 1	×	×
2 <i>p</i> 2		
3 <i>p</i> 3		

The main theor

Definitions
Well-definedness

On bounding the

Ternary case

The main theore

The main theo

The proof

Some notation

Propositions 4

Propositions 4–5
Putting everything together

Putting everything together

	$f_1(w)$	$f_2(w)$
0 <i>p</i> 0	×	×
1 <i>p</i> 1	×	×
2 <i>p</i> 2	×	
3 <i>p</i> 3		

Introduction

Definitions
Some interesti

Some interesting roles of subpalindromes

The main theor

n-ary case
Definitions
Well-definedness

On bounding the

Ternary case

The main theore

case

The main theore

Some notation

Proposition 3

Propositions 4-

Putting everything together

Putting everything together

	$f_1(w)$	$f_2(w)$
0 <i>p</i> 0	×	×
1 <i>p</i> 1	×	×
2 <i>p</i> 2	×	
3 <i>p</i> 3	×	

Putting everything together

Putting everything together

	$f_1(w)$	$f_2(w)$
0 <i>p</i> 0	×	×
1 <i>p</i> 1	×	×
2 <i>p</i> 2	×	
3 <i>p</i> 3	×	
	✓	

Putting everything together

Putting everything together

	$f_1(w)$	$f_2(w)$
0 <i>p</i> 0	×	×
1 <i>p</i> 1	×	×
2 <i>p</i> 2	×	
3 <i>p</i> 3	\in	

The main theo

Definitions
Well-definedness f

On bounding the MP-ratio from above

Ternary case

The main theore

The main theor

The proof

Some notation

Propositions 4-

Putting everything together

Putting everything together

	$f_1(w)$	$f_2(w)$
0 <i>p</i> 0	×	×
1 <i>p</i> 1	×	×
2 <i>p</i> 2	×	
3 <i>p</i> 3	\in	×

Introduction

Definitions

Some interesting roles of subpalindromes

The main theor

Definitions
Well-definedness f

On bounding the MP-ratio from abov

Ternary case

The main theor

The main theor

The proof

Some notation Proposition 3

Propositions 4-

Putting everything together

Putting everything together

	$f_1(w)$	$f_2(w)$
0 <i>p</i> 0	×	×
1 <i>p</i> 1	×	×
2 <i>p</i> 2	×	×
3 <i>p</i> 3	\in	×

The main theo

Definitions
Well-definedness

On bounding the

The main theore

The main theore

case

The proof

Some notation

Propositions 4

Putting everything together

Putting everything together

	$f_1(w)$	$f_2(w)$
0 <i>p</i> 0	×	×
1 <i>p</i> 1	×	×
2 <i>p</i> 2	×	×
3 <i>p</i> 3	\in	×
		\checkmark