

Clustering of return words in languages of Interval Exchange Transformations

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WORDS 2025

Nancy, 2 juillet 2025



louche – chelou

louche – chelou

fou – ouf

louche – chelou

fou – ouf

Two words uv and vu are *conjugates*.

$$[w] = \{w' = uv \mid w = vu\}$$

The smallest element in $[w]$ (w primitive) is a *Lyndon word*.

Burrows-Wheeler Transform

Let $(\mathcal{A}, <)$ be an ordered alphabet and $w \in \mathcal{A}^*$.

b a n a n a



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b	a	n	a	n	a						
	a	n	a	n	a	b					
		n	a	n	a	b	a				
			a	n	a	b	a	n			
				n	a	b	a	n	a		
					a	b	a	n	a	n	



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$$\text{bwt}_{\mathcal{A}}(\text{banana}) = \text{nnbaaa}$$



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Proposition

w is a conjugate of u^p if and only if $\text{bwt}_{\mathcal{A}}(u) = b_1 \dots b_n$ and $\text{bwt}_{\mathcal{A}}(w) = b_1^p \dots b_n^p$.

$$\text{bwt}_{\mathcal{A}}(\text{bon}) = \text{nob} \quad \text{bwt}_{\mathcal{A}}(\text{bonbon}) = \text{nnoobb}$$

Clustering words

A word $w \in \mathcal{A}^*$ is π -clustering for $\mathcal{A}$ if $\text{bwt}_{\mathcal{A}}(w) = a_{\pi(1)}^{k_1} \cdots a_{\pi(d)}^{k_d}$, where $k_i = |w|_{a_{\pi(i)}}$.

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$$\mathcal{A} = \{a < b < n\},$$

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$$\mathcal{A}'' = \{n < a < b\}$$

$$\text{bwt}_{\mathcal{A}}(\text{banana}) = \text{nnbaaa},$$

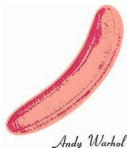
$$\text{bwt}_{\mathcal{A}'}(\text{banana}) = \text{bnnaaa},$$

$$\text{bwt}_{\mathcal{A}''}(\text{banana}) = \text{aabnna}$$

$$\pi = \begin{pmatrix} a & b & n \\ n & b & a \end{pmatrix}$$

$$\pi' = \begin{pmatrix} a & n & b \\ b & n & a \end{pmatrix}$$

not clustering



Clustering words

Proposition

Let $w = u^p \in \mathcal{A}^*$. Then w is π -clustering for $\mathcal{A}$ if and only if u is π -clustering for $\mathcal{A}$.

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Theorem [Mantaci, Restivo, Sciortino (2003)]

Over a binary alphabet (perfectly) clustering words are exactly powers of *Christoffel words* (i.e., *finite standard Sturmian words*) and their conjugates.

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And over larger alphabets ?

Clustering words

A characterization with palindromes : No Lemon, No Melon

Theorem [Lapointe, Reutenauer (2024)]

Let $w \in \mathcal{A}^*$ be a primitive word. The following are equivalent :

- w is a perfectly clustering Lyndon word
- $$\begin{cases} w = a_1 p_1 a_2 p_2 \cdots p_{d-1} a_d, & p_i = \tilde{p}_i \quad (\text{special factorization}) \\ w = q_1 q_2, & q_j = \tilde{q}_j \end{cases}$$
- $$\begin{cases} w = a_1 p_1 a_2 p_2 \cdots p_{d-1} a_d \\ [w] = [a_d p_{d-1} \cdots p_2 a_2 p_1 a_1] \end{cases}$$

$$\text{bwt}_{\mathcal{A}}(\text{ananas}) = n^2 s^1 a^3$$

$$\begin{array}{l} a \cdot \varepsilon \cdot n \cdot \text{ana} \cdot s \\ \text{anana} \cdot s \end{array}$$

$$[a \varepsilon n \text{ana} s] = [s \text{ana} n \varepsilon a]$$

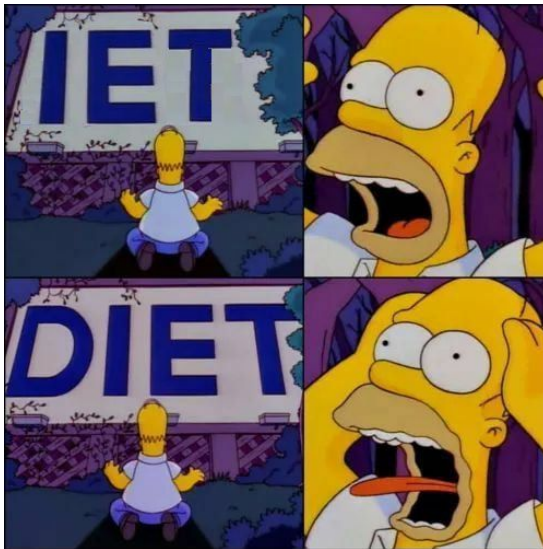
Clustering words and (D) IETs

Theorem [Ferenczi, Zamboni (2013)]

Let $w \in \mathcal{A}^*$ be primitive, with $\text{Card}(\mathcal{A}) = d$. The following are equivalent :

1. w is π -clustering ;
2. ww occurs in the trajectory of a *minimal d -DIET* with permutation π ;
3. ww occurs in the trajectory of a *d -DIET* with permutation π ;
4. ww occurs in the trajectory of a *d -IET* with permutation π .

IETs ? DIETs ?

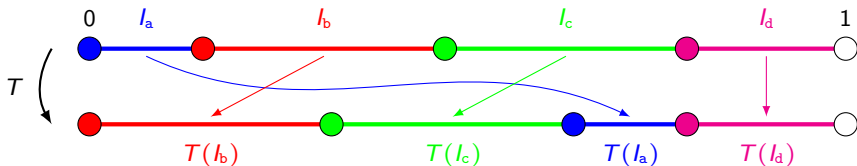


Interval exchanges

Let $(I_a)_{a \in \mathcal{A}}$ be a partition of $[\ell, r]$.

An *interval exchange transformation* (IET) is a map $T : [\ell, r] \rightarrow [\ell, r]$ defined by

$$T(z) = z + \tau_a \quad \text{if } z \in I_a.$$

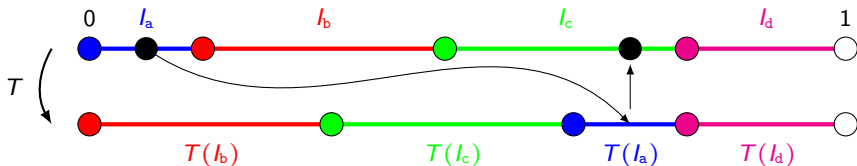


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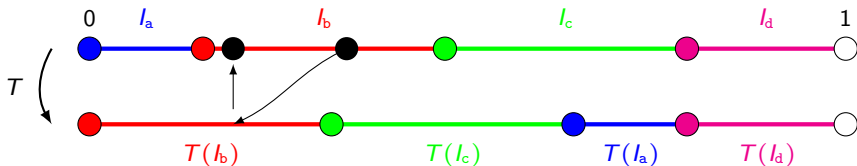


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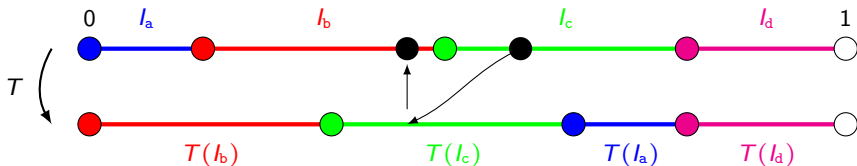


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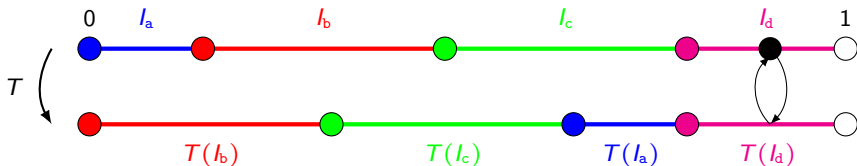


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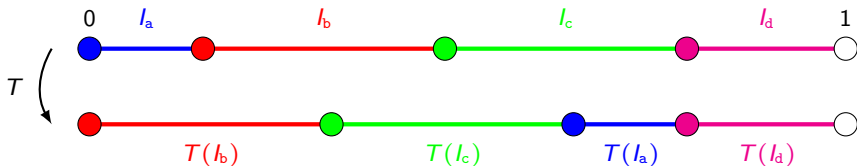


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$$\mathcal{A} = \{a < b < c < d\} \qquad \pi = \begin{pmatrix} a & b & c & d \\ b & c & a & d \end{pmatrix}$$

Minimality and regularity

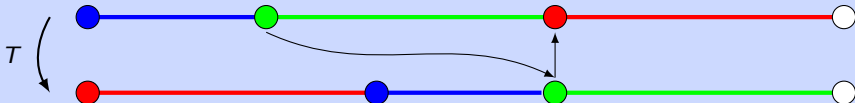
T is *minimal* if for any $z \in [\ell, r)$ the orbit $\mathcal{O}(z) = \{T^n(z) \mid n \in \mathbb{Z}\}$ is dense in $[\ell, r)$.

T is *regular* if the orbits of the non-zero formal discontinuities are infinite and disjoint.

Theorem [Keane (1975)]

A regular interval exchange transformation is minimal.

Example (the converse is not true)

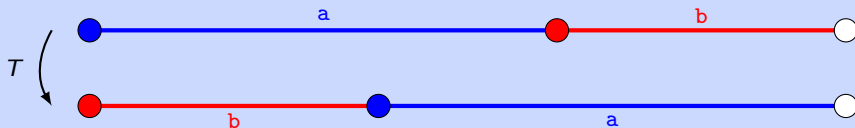


Trajectories

The *trajectory* of $z \in [\ell, r)$ under T is the infinite word $\Omega_T(z) = w_0 w_1 \dots \in \mathcal{A}^\omega$ defined by

$$w_n = a \quad \text{if } T^n(z) \in I_a.$$

Example (Fibonacci)

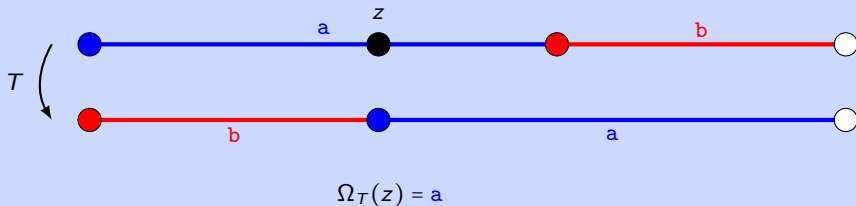


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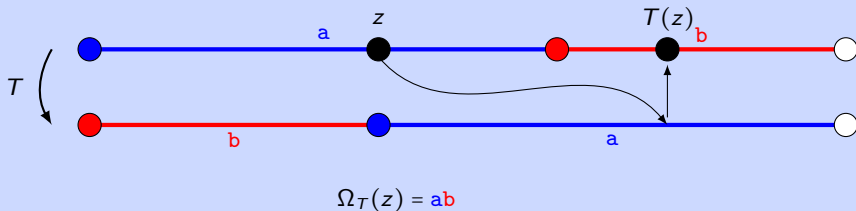


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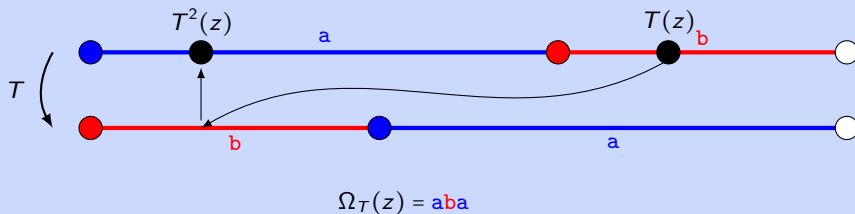


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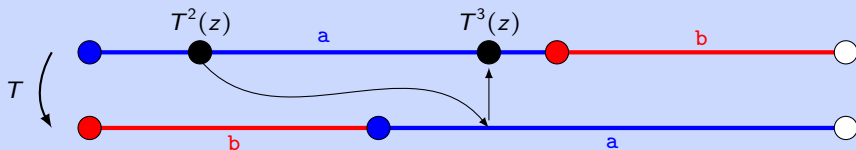


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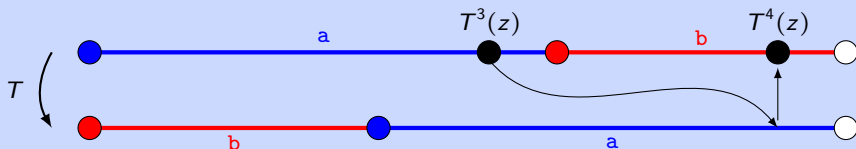
$$\Omega_T(z) = abaa$$

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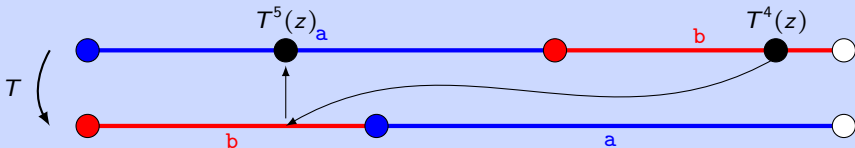
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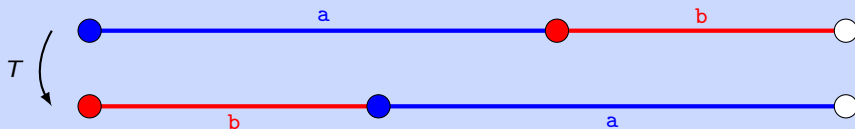


$$\Omega_T(z) = \text{abaaba}\cdots$$

Interval exchange languages

The set $\mathcal{L}(T) = \bigcup_{z \in [\ell, r)} \mathcal{L}(\Omega_T(z))$ is a (*minimal, regular*) *interval exchange language*.

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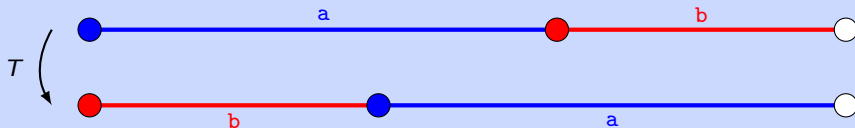


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Remark. When T is minimal, $\mathcal{L}(\Omega_T(z))$ does not depend on the point z .

Example (Fibonacci)

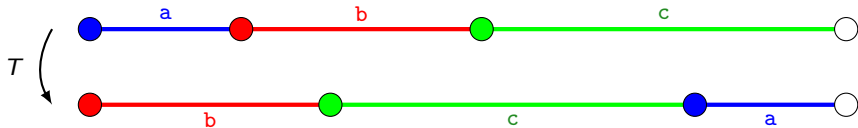


$$\mathcal{L}(T) = \{\varepsilon, \textcolor{blue}{a}, \textcolor{red}{b}, \textcolor{blue}{aa}, \textcolor{blue}{ab}, \textcolor{red}{ba}, \textcolor{blue}{aab}, \textcolor{red}{aba}, \textcolor{red}{baa}, \textcolor{red}{bab}, \textcolor{blue}{aaba}, \dots\}$$

From letters to words

Given a IET T and a word $w = w_0 w_1 \dots w_m \in \mathcal{A}^*$, let

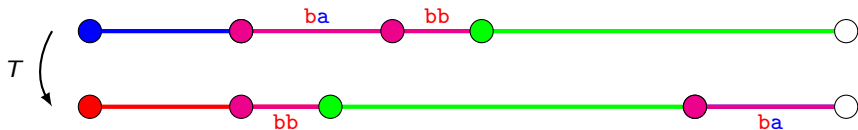
$$I_w = I_{w_0} \cap T^{-1}(I_{w_1}) \cap \dots \cap T^{-m}(I_{w_m}) \quad (\text{by convention } I_\varepsilon = [\ell, r))$$



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$$I_{ba} = I_b \cap T^{-1}(I_a)$$

$$T^2(I_{ba}) = T^2(I_b) \cap T(I_a)$$

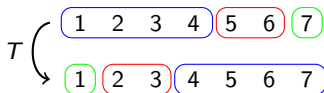
Thus $\Omega_T(z)$ starts with w for every $z \in I_w$.

From IETs to DIETs

Let $(n_1, n_2, \dots, n_d)$ be a composition of $n = \sum n_i$.

A *discrete interval exchange transformation* (DIET) is a map $T : \mathbb{N}_n \rightarrow \mathbb{N}_n$ defined by

$$T(k) = k + t_i \quad \text{if } k \in \left[\sum_{j < i} n_j, \sum_{j \leq i} n_j \right).$$



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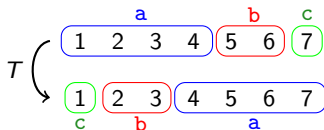
$$\Omega_T(1) = 1\,4\,7\,1\,4\,7\,1\,4\,7\,\dots = (1\,4\,7)^\omega$$

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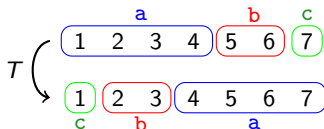
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$$\Omega_T(1) = a a c a a c a a c \cdots = (a a c)^\omega$$

$$\mathcal{L}(T) = \mathcal{L}((a a c)^\omega \cup (a b)^\omega \cup (a b)^\omega)$$

Return words

Question [Lapointe (2021)]

Are return words of a symmetric IET perfectly clustering words?

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$$\mathcal{R}_{\mathcal{L}}(u) = \{w \in \mathcal{A}^* \mid uw \in (\mathcal{L} \cap \mathcal{A}^* u) \setminus \mathcal{A}^+ u \mathcal{A}^+\}$$

Example (Fibonacci)

$\mathbf{f} = \underline{a} \underline{b} a \underline{a} \underline{b} a \underline{b} a \underline{a} \underline{b} a \underline{a} \underline{b} a \underline{b} a \underline{a} \underline{b} a \underline{a} \underline{b} a \dots$

$$\mathcal{R}(\underline{a} \underline{b} a) = \{\underline{a} \underline{b} a, \underline{b} a\}$$



Induced transformations

Let T be a minimal IET and $J \subset [\ell, r)$.

The *transformation induced* by T (*first return map* of T) on J is $T' : J \rightarrow J$ defined by

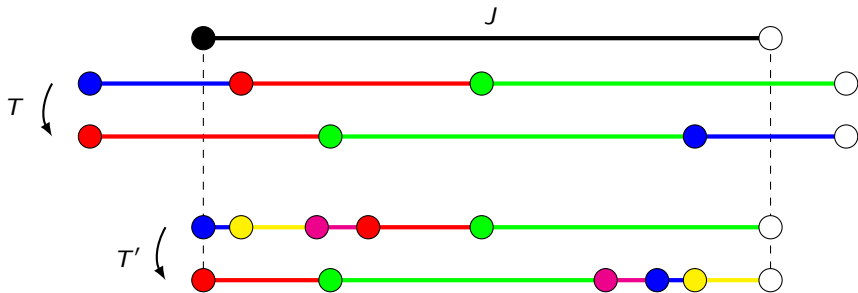
$$T'(z) = T^{\nu(z)}(z) \quad \text{with} \quad \nu(z) = \min\{n > 0 \mid T^n(z) \in J\}$$

Induced transformations

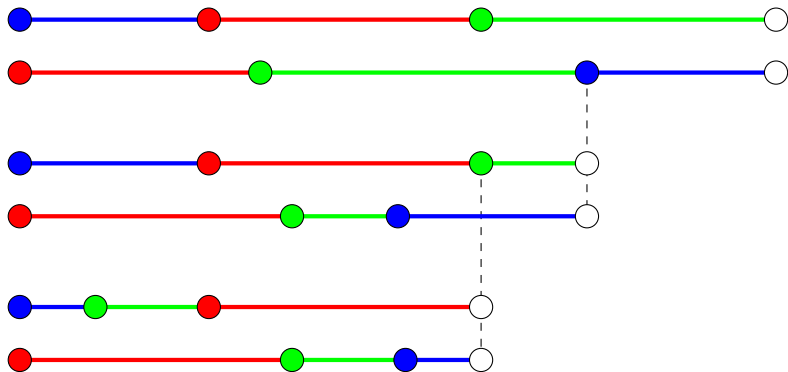
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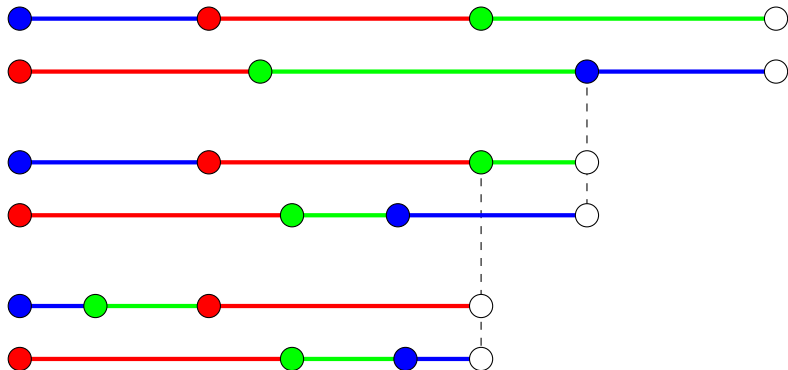
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Right Rauzy induction



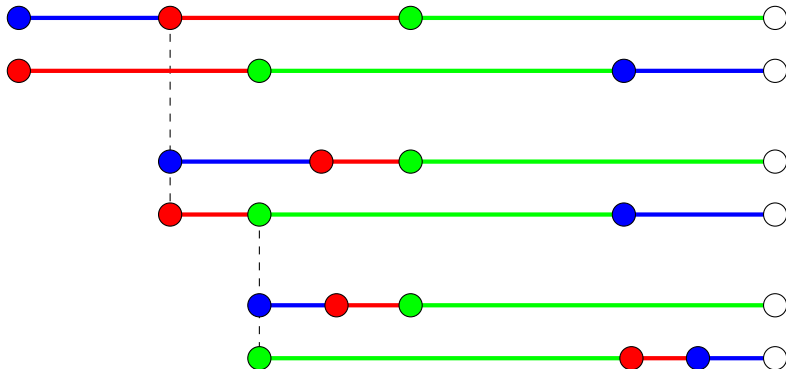
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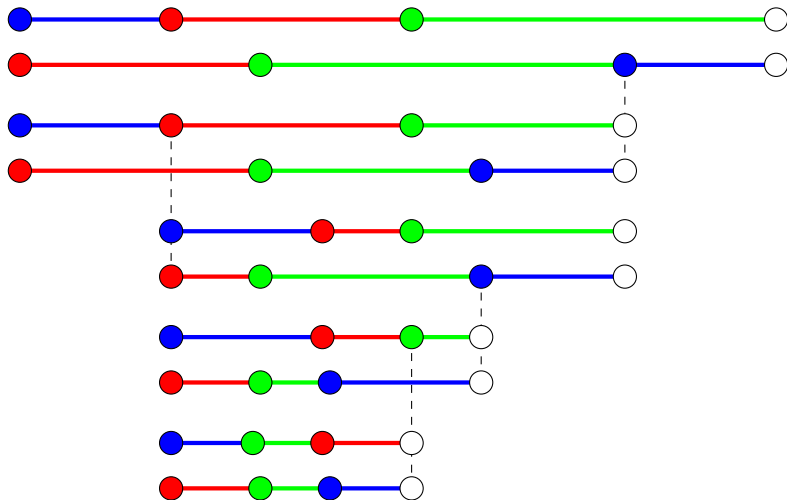
Theorem [Rauzy (1979)]

If T is regular, the right Rauzy induced transformation is regular on the same alphabet.

Left Rauzy induction



Two-sided Rauzy induction



Rauzy induction on I_w

Theorem [D., Perrin (2017)]

Let T be a regular IET and $w \in \mathcal{L}(T)$.

The regular IET T' induced on I_w can be obtained by two-sided Rauzy induction.

Rauzy induction on I_w

Theorem [D., Perrin (2017)]

Let T be a regular IET and $w \in \mathcal{L}(T)$.

The regular IET T' induced on I_w can be obtained by two-sided Rauzy induction.

Moreover, each step is associated with an automorphism of $F_{\mathcal{A}}$ of the form

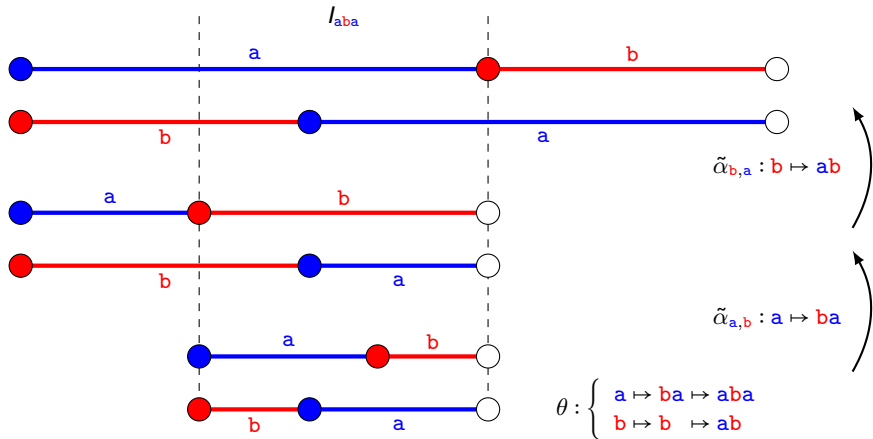
$$\alpha_{a,b} = \begin{cases} a \mapsto ab \\ c \mapsto c \quad \forall c \neq a \end{cases} \quad \text{or} \quad \tilde{\alpha}_{a,b} = \begin{cases} a \mapsto ba \\ c \mapsto c \quad \forall c \neq a \end{cases}$$

and for every $z \in I_w$

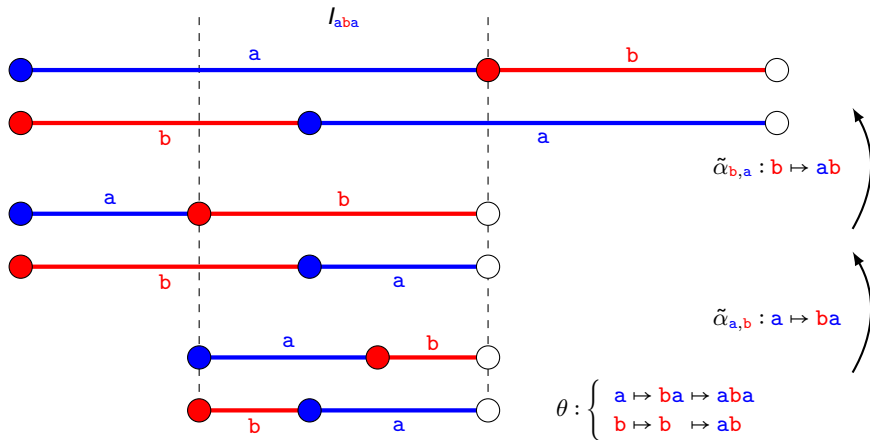
$$\Omega_T(z) = \theta(\Omega_{T'}(z)).$$

(θ is obtained from the morphisms above, but backwards!)

Rauzy induction on I_w



Rauzy induction on l_w



$$\mathcal{R}(aba) = \{aba, ab\}$$

Return words in IETs are clustering
a.k.a. the main result

Corollary

$$\mathcal{R}(w) = \theta(\mathcal{A}).$$

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Trivial fact

Letters are clustering.

Return words in IETs are clustering

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Corollary

$$\mathcal{R}(w) = \theta(\mathcal{A}).$$

Trivial fact

Letters are clustering.

Proposition (way too technical to be stated properly)

Under the "right conditions" each $\alpha_{a,b}$ (resp., $\tilde{\alpha}_{a,b}$) preserves clustering.

Return words in IETs are clustering

a.k.a. the main result

Corollary

$$\mathcal{R}(w) = \theta(\mathcal{A}).$$

Trivial fact

Letters are clustering.

Proposition (way too technical to be stated properly)

Under the "right conditions" each $\alpha_{a,b}$ (resp., $\tilde{\alpha}_{a,b}$) preserves clustering.

Theorem

Let T be regular IET and $u \in \mathcal{L}(T)$. Each $w \in \mathcal{R}(u)$ is clustering.



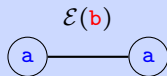
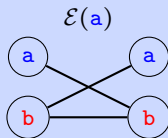
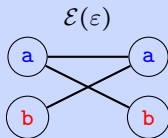
Extension graphs

The *extension graph* of $w \in \mathcal{L}$ is the bipartite graph $\mathcal{E}(w)$ with vertices

$$L(w) = \{a \in \mathcal{A} \mid aw \in \mathcal{L}\} \quad \text{and} \quad R(w) = \{b \in \mathcal{A} \mid wb \in \mathcal{L}\},$$

and edges $B(w) = \{(a, b) \in \mathcal{A} \times \mathcal{A} \mid awb \in \mathcal{L}\}$.

Example (Fibonacci, $\mathcal{L} = \{\varepsilon, a, b, aa, ab, ba, aba, baa, bab, \dots\}$)



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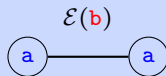
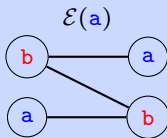
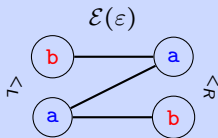
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and edges $B(w) = \{(a, b) \in \mathcal{A} \times \mathcal{A} \mid awb \in \mathcal{L}\}$.

$\mathcal{E}(w)$ is *compatible* with two orders $<_L, <_R$ on $\mathcal{A}$ if for every $(a, b), (c, d) \in B(w)$

$$a <_L c \quad \Rightarrow \quad b \leq_R d.$$

Example (Fibonacci, $b <_L a$, and $a <_R b$)



See the forest for the IETs
It's all Greek to me!

$\mathcal{L}$ is *dendric* (δένδρον) if every $\mathcal{E}(w)$ is a tree. It is *alsinic* (ἄλσος) if $\mathcal{E}(w)$ is a forest.

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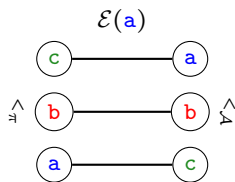
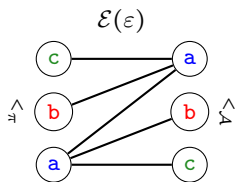
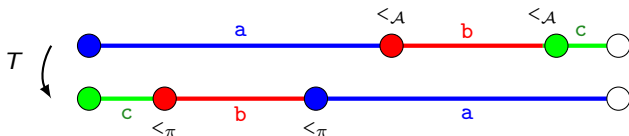
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Theorem [Ferenczi, Zamboni (2008) ; Ferenczi, Hubert, Zamboni (2024)]

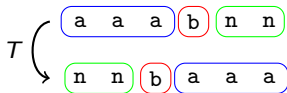
- $\mathcal{L}$ is an interval exchange language **iff** it is (uniformly) recurrent ordered alsinic.
- $\mathcal{L}$ is a minimal interval exchange language **iff** it is aperiodic (uniformly) recurrent ordered alsinic.
- $\mathcal{L}$ is a regular interval exchange language **iff** it is (uniformly) recurrent ordered dendric.

See the forest for the IETs





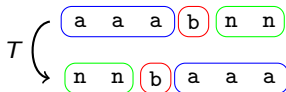
See the forest for the DIETs



$\text{bwt}_{\{a < b < n\}}(\text{banana}) = \text{nnb}aaa$



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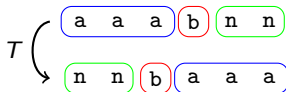
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Theorem [Ferenczi, Hubert, Zamboni (2023)]

A word $w \in \mathcal{A}^*$ is π -clustering **if and only if** for every (bispecial) word $v \in \mathcal{L}(w^\omega)$ the graph $\mathcal{E}(v)$ is compatible with $<_{\mathcal{A}}$ and $<_{\pi}$.



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Corollary

A word $w \in \mathcal{A}^*$ is π -clustering **if and only if** $\mathcal{L}(w^\omega)$ is ordered alsinic for $<_{\mathcal{A}}$ and $<_{\pi}$.

Where to go from here ?

- What about the permutation π ?

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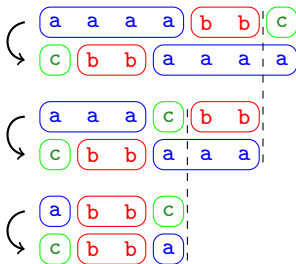
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$$\mathcal{L}((\text{aac})^\omega \cup (\text{ab})^\omega \cup (\text{ab})^\omega)$$

$$\theta : \begin{cases} a \mapsto a \mapsto a \\ b \mapsto ab \mapsto ab \\ c \mapsto c \mapsto ac \end{cases}$$

$$\mathcal{R}(a) = \{a, ab, ac\}$$

Cimer coupbeau !

