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Open problem session

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Global complexity of billiard inside a cube

Consider the billiard map in the cube coded with 3 letters: the same letter for parallel faces.
The (global) cubic billiard language is the set of all the words which code any finite billiard trajectory.

Can we find an exact formula for its complexity function?

We know that it is in $\Theta(n^6)$.
For the same problem in the square, we have an exact formula.

Štěpán Holub

Periodicity forcing

Find all four-tuples $(i, j, k, \ell) \in \mathbb{N}^4$ for which the following holds:

If X, Y, U, V are four words such that $|X|, |Y|, |U| \leq |V|$ satisfying

$$X^i Y X^j Y X^k Y X^\ell = U^i V U^j V U^k V U^\ell$$

then all four words commute or $X = U$ and $Y = V$.
(we say that the equality is *periodicity forcing*).

Note that this can be naturally reformulated as follows:
For what words

$$a^i b a^j b a^k b a^\ell$$

are there two distinct and not (both) periodic morphisms g and h such that

$$g(a^i b a^j b a^k b a^\ell) = h(a^i b a^j b a^k b a^\ell).$$

(Such a word is called *binary equality word*. The length condition above breaks the symmetry between a and b — as well as between g and h — by requiring that the longest word is $h(b)$.)

Some known cases:

- $YXYYYX = VUVVVU$ is periodicity forcing
- $XYXYYY = UVUVVV$ is periodicity forcing
- $X^iYYY = U^iVVV$ is not periodicity forcing
- $XYXYXYX = UVUVUVU$ and $YXYXY = VUVUV$ are not periodicity forcing

Simplest unknown case:

$$XXYYYXYX = UUVVVUVU.$$

Gandhar Joshi

Anti-recurrence sequences

The following problem stems from our recent preprint [2]. In a linear recurrence sequence, each term is a linear combination of the ones that came before it. The first example that jumps to everyone's mind is the Fibonacci sequence

$$F_{n+1} = F_n + F_{n-1},$$

starting from the initial conditions $F_0 = 0$ and $F_1 = 1$. Recurrence sequences are defined by earlier terms in the sequence. In contrast with this, the *anti-recurrence* sequences are defined by earlier terms that are *not* in the sequence. The anti-Fibonacci numbers start with $A_0 = 0$. They extend by the rule that the next anti-Fibonacci number is the sum of the two most recent NON-members of the anti-Fibonacci sequence. The first two non-members 1 and 2 add up to the anti-Fibonacci $A_1 = 3$. The next two non-members are 4 and 5, which add up to the anti-Fibonacci $A_2 = 9$, etc. This sequence is listed under A075326 in the On-Line Encyclopedia of Integer Sequences.

$$0, 3, 9, 13, 18, 23, 29, 33, 39, 43, 49, 53, 58, 63, 69, 73, 78, 83, 89, 93, 98, 103, 109, 113, \ldots$$

All numbers with final digit 3 are anti-Fibonaccis, and the other anti-Fibonaccis either end with a 9 or an 8. Hofstadter in an unpublished note [3] observed, without giving a proof, that the pattern of 9's and 8's can be generated from a period-doubling substitution

$$9 \mapsto 98, \;\; 8 \mapsto 99.$$

The proof was supplied by Thomas Zaslavsky, in another unpublished note [6]. In particular, he gave an explicit equation for the anti-Fibonacci numbers

$$\text{For all } n \geq 1, \quad \text{A075326}(n) - 5n + 2 = \texttt{PD}_{n-1}. \tag{1}$$

Note that the indexing for $\texttt{PD}_n$ runs from 0 and not from 1.

Clark Kimberling and Peter Moses studied the more general class of complementary sequences [4], for which anti-recurrence sequences are a special case. They observed some properties of anti-recurrence sequences, which Kimberling entered as conjectures under A265389, A299409, A304499, and A304502 in the OEIS. The conjectures for the first two sequences were verified by Bosma et al. [1] using Hamoon Mousavi's automatic theorem prover `Walnut` [5]. We settled the other two conjectures on A304499 and A304502 in [2], again with the assistance of `Walnut`. These conjectures can be combined into a meta-conjecture, which specifies the discussion in section six of [4]. It was named the Clergyman's conjecture in [1].

Conjecture 1. Every anti-recurrence sequence is a sum of a linear sequence and an automatic sequence.

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Anuran Maity

Closed-rich constant of infinite words

ON THE CLOSED-RICH CONSTANT OF INFINITE WORDS

ANURAN MAITY; SVETLANA PUZYNINA

A finite word is called *closed* if it has length at most 1 or it contains a proper factor that occurs both as a prefix and as a suffix but does not have internal occurrences. For example, the word *ababa* is a closed word. The number of closed factors in a word w is denoted by $\texttt{Cl}(w)$. An infinite word u is called closed-rich ([1]) if there exists a constant $C > 0$ such that any factor of length n of u contains at least Cn^2 distinct closed factors. The real number $C_u = \sup\{C : \texttt{Cl}(w) \geq C|w|^2 \text{ for each } w \in \Sigma^+ \cap Fac(u)\}$ is called the *closed-rich constant* of u. Equivalently, $C_u = \inf\{\frac{\texttt{Cl}(w)}{|w|^2} \mid w \in Fac(u), |w| \geq 1\}$. Consider

$$C_{sup} = \sup\{C_u : \ u \text{ is an infinite closed-rich word}\}.$$

In our work, we have shown that $0.0952 < C_{sup} \leq 0.165964$. With the help of infinite Fibonacci sequence f, we get the lower bound of C_{sup}. This work leads to the following question and conjectures:

- **Question 1 :** Is it true that any real number in $(0, C_{sup})$ is a closed-rich constant of some infinite closed-rich word? If not, is the set of closed-rich constants dense in $(0, C_{sup})$?

- **Conjecture 1:**

  For the Fibonacci sequence f, let $M_n = \min\{\mathtt{Cl}(w) : w \text{ is a factor of } f \text{ of length } n\}$ and $R_n = M_n - M_{n-1}$. Then,

$$\{R_n\}_{n\geq 1} = 1, 1, 1, 1, \underbrace{F_0, F_2, F_0, F_2}, \underbrace{F_1, F_1, F_3, F_1, F_1, F_3}, \underbrace{F_2, F_2, F_2, F_4, F_4, F_2, F_2, F_2, F_4, F_4}, \cdots$$

  In other words, if we denote repetitions in a sequence by exponents and concatenation by the symbol $\prod$, then we can rephrase the above statement as follows:

$$\{R_n\}_{n\geq 1} = 1, 1, 1, 1, \prod_{m=0}^{\infty} {F_m}^{F_m}, {F_{m+2}}^{F_{m-1}}, {F_m}^{F_m}, {F_{m+2}}^{F_{m-1}}.$$

- **Conjecture 2:**

  For the Fibonacci sequence f, the closed-rich constant $C_f = \frac{5\phi+3}{45\phi+29} \approx 0.10893$ where ϕ is the golden ratio.

[Recall the Fibonacci sequence: Let $(f_n)_{n\geq -1}$ denotes the sequence of Fibonacci words where $f_{-1} = b$, $f_0 = a$, and $f_n = f_{n-1}f_{n-2}$ for $n \geq 1$. The word f_n are referred to as the n-th Fibonacci word and $f = \lim_{n\to\infty} f_n$ is called the infinite Fibonacci word. Also, $|f_n| = F_n$ for $n \geq -1$.]

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Kristina Ago

MP-ratio for n-ary words

The *MP-ratio* is one of measures of how palindromic a given word is. For an n-ary word w, its MP-ratio is defined as $\frac{|rws|}{|w|}$, where (r, s) is a pair words of minimal length such that rws is *minimal-palindromic* (contains no palindromic subwords longer than $\lceil \frac{|rws|}{n} \rceil$). This notion was first introduced by Holub and Saari [2], in the setting of binary words.

Recent work has established optimal upper bounds for small alphabets: the MP-ratio is at most 4 for binary words, 6 for ternary words, and 8 for quaternary words. All these bound are optimal in the asymptotic sense. Also, it is known that in the n-ary case the optimal bound is between $2n$ and the order of the growth $n2^{\frac{n}{2}}$. For a more comprehensive overview, we refer to the introduction of the paper from this conference [1].

Open problem

What is the optimal upper bound on the MP-ratio for n-ary words when $n \geqslant 5$? Of course, obtaining the exact formula would be the ultimate goal. If not, then at least the asymptotic behavior would still be very satisfying. If this is still too hard, then replacing $2n$ by anything superlinear, and/or finding any $o(n2^{\frac{n}{2}})$ upper bound could also be considered as a significant step forward.

A (possibly) more approachable subproblem

The general question seems quite challenging (all the mentioned versions of it). A more modest goal would be to show that, for $n = 5$, the optimal upper bound on the MP-ratio is strictly greater than 10. This means that there exists a word w that does not possess an MP-extension (r, s) such that $|rws| \leqslant 10|w|$. Let us try to explain why our intuition suggests that this might be true.

For alphabets with up to four letters, existing constructions rely heavily on very uneven (asymmetric) arrangements of the letters. For example, in the case of four letters, the letters 0 and 1 are placed only on one side, while the letters 2 and 3 are arranged as asymmetrically as possible around w. Our experiments suggest that this is indeed the key for any reasonable construction. However, when the alphabet size increases to five letters, breaking symmetry in this way is not enough; no matter how the fifth letter is arranged, it tends to create too long palindromic subwords.

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Jarosław Grytczuk

Square-free words on a chessboard

Let n be a positive integer. Consider a $3 \times n$ chessboard whose squares have been filled with letters so that each column contains three distinct letters.

Is it always possible to permute the letters in each column so that a square-free word is formed in each row?

Martin Lustig

Substitutive subshifts

4. What really is the true nature of a subshift ?

The free monoid $\mathcal{A}^*$ over the alphabet $\mathcal{A} = \{a_1, \dots, a_d\}$ embeds canonically into the free group $F(\mathcal{A})$ over $\mathcal{A}$. But contrary to $\mathcal{A}^*$, where the minimal generating system $\mathcal{A}$ is uniquely determined by $\mathcal{A}^*$, in $F(\mathcal{A})$ there are infinitely many sets $\mathcal{B} = \{w_1, \dots, w_d\} \subseteq F(\mathcal{A})$ with canonical isomorphisms $F(\mathcal{B}) \cong F(\mathcal{A}) \cong F_d$, and none of these *bases* for the free group F_d of *rank* $d \geqslant 2$ is preferred in any way. Any subshift X over $\mathcal{A}$ gives canonically rise (by passing to the language $\mathcal{L}(X)$) to a "subshift with inverses" over $\mathcal{B}$, which has led to the basis-free notion of an *algebraic lamination* in F_d (see [8]), together with a canonical embedding $\Sigma(\mathcal{A}) \to \Lambda(F(\mathcal{A}))$ of the space of subshifts $\Sigma(\mathcal{A})$ into the space of algebraic laminations $\Lambda(F(\mathcal{A}))$. Similarly, the space of invariant measures $\mathcal{M}(\mathcal{A}^{\mathbb{Z}})$ embeds canonically into the space of *currents* $\mathcal{M}(F(\mathcal{A}))$.

ASIDE: Symbolic dynamists feel traditionally uneasy about the behavior of inverses under morphisms, but this is mainly due to the fact that the notion of *train track maps* has not yet dissipated into the symbolic dynamics community. With this tool the whole S-adic machinery as well as most other symbolic dynamics methods and results could (and should) be carried over from symbolic dynamics to geometric group theory.

There is also a "response" from geometric group theory towards symbolic dynamics, namely that any property of a subshift which is not invariant under change of basis is not accepted as "intrinsic" property, just like properties of matrix groups are not intrinsic properties of the group in question if they are not invariant under group isomorphisms, or similarly for properties of topological objects, if they depend on the embedding of the object in an ambient space (like the well known "2-sided coloring" criterion for a surface to be orientable or not).

To stay within symbolic dynamics terminology I'd like to make this a bit more precise:

Definition 4.1. A property of a subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is said to be *intrinsic* if, first, for any monoid morphisms $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ which is recognizable in X, the property must also hold for the subshift $\sigma(X) \subseteq \mathcal{B}^{\mathbb{Z}}$. Second, for any monoid morphism $\sigma' : \mathcal{C}^* \to \mathcal{A}^*$ and any subshift $Y \subseteq \mathcal{C}^{\mathbb{Z}}$ with $\sigma'(Y) = X$, if σ' is recognizable in Y, then the property must also hold for Y.

Examples of intrinsic properties are minimality, unique ergodicity, the number $e(X)$ of ergodic probability measures, the statements $h_X = 0$ or $h_X > 0$, and the growth-type of the complexity function $p_X(\cdot)$. The value of $h_X > 0$ however is *not* intrinsic, and neither is the complexity function itself (and not even its equivalence class $\Theta(p_X)$, see [10]).

5. Classification of substitutive subshifts

The speaker has very recently discovered (in the context of investigating a certain type of free group automorphisms) a new computable invariant for any minimal substitutive subshift, which consists of a cyclic sequence of finite graphs and graph maps between them. This invariant appears to be (work in progress) a characterizing invariant of the given subshift, up to recognizable morphisms as in Definition 4.1.

The technicalities of the graphs in question are not yet matured enough to be presented here (other than via the examples given in the Annex below), but the main idea ought to be conveyed anyway:

For any $n \geqslant 0$ the *level* $2n$ *Rauzy graph* $R_{2n}(X)$ of a subshift $X \subseteq \mathcal{A}$ can be reinterpreted as obtained in the following way: One first realizes X graphically as a (typically infinite) collection of lines $\gamma(\mathbf{x})$, one for every $\mathbf{x} \in X$, subdivided as biinfinite edge path, with edges labeled by letters from $\mathcal{A}$ according to the letters x_k on $\mathbf{x} = \ldots x_{-1}x_0x_1\ldots$. In a second step we identify any two vertices $P \in \gamma(\mathbf{x})$ and $Q \in \gamma(\mathbf{x}')$ iff the finite sub-edge-paths of length $2n$ on $\mathbf{x}$ and on $\mathbf{x}'$, centered at P and Q respectively, read off the same word. Finally, to get the finite graph $R_{2n}(X)$ we need to identify any two edges with same endpoints and same label. The subshift X can then be read off from suitable edge paths in $\mathbb{R}_{2n}(X)$, and X is characterized by the fact that this "read-off property" holds for any $n \geqslant 1$. Another pay-off of this alternative construction are canonical label-preserving graph morphisms $R_{2m}(X) \to R_{2n}(X)$ for any $m \geqslant n \geqslant 0$, which define in turn a canonical S-adic *Rauzy development* of X which is always totally recognizable. Locally, each of these label-preserving graph morphisms is a composition of vertex-identifications and edge-foldings.

In the special case where X is the subshift generated by a primitive substitution σ, we can use the incidence matrix $M(\sigma)$ and one of its (left) row PF-eigenvectors $\vec{v}^*$, in order to define from the coefficients of $\vec{v}^*$ a length function L on the letters of $\mathcal{A}$ (and thus by summation on all of $\mathcal{A}^*$) which satisfies $L(\sigma(a_i)) = \lambda L(a_i)$ for each $a_i \in \mathcal{A}$, where $\lambda > 1$ is the PF-eigenvalue of $M(\sigma)$.

We can now repeat the above definition of the Rauzy graph $R_{2n}(X)$, but replace the combinatorial length used there (when considering for the vertex identification the $2n$-length sub-edge-paths) by the length L, and apply the identification device not just to vertices but also to points in the interior of edges (with "read-off equality of sub-edge-paths" refined by passing to σ-iterates of those paths).

This gives a continuity of graphs $R(\vec{v}^*)$, one for any positive eigenvector $\vec{v}^*$ within the uniquely determined PF-eigen-direction of $M(\sigma)$, and for $\vec{v}_2^* = \lambda\vec{v}_1^*$ the corresponding graphs are related by a graph isomorphism $R(\vec{v}_1^*) \to R(\vec{v}_2^*)$ which stretches every edge by the factor λ. The composition $R(\vec{v}_1^*) \to R(\vec{v}_2^*) \to R(\vec{v}_1^*)$ of this homothetic graph isomorphism with the above "vertex-identification & edge-folding" map $R(\vec{v}_2^*) \to R(\vec{v}_1^*)$ is a topological realization of some substitution σ' which generates X.

To get to the desired finite cycle of graphs we have to discretize the just obtained continuous "loop of graphs" in a canonical way. There are several possibilities for this canonical discretization (all built on periodic points of the composed maps $R(\vec{v}_1^*) \to R(\vec{v}_2^*) \to R(\vec{v}_1^*)$, see the examples in the Annex below), and the choice of the most natural among them is one of the problems still on my desk.

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Anna Frid

Palindromic length in the free group

The palindromic length of a finite word is the minimal number of palindromes needed to construct it. Clearly, this notion is different in the free semigroup and in the free group: for example, in the semigroup, the palindromic length of $abca$ is 4, and in the group, it is 3 since $abca = (aba)(a^{-1}a^{-1})(aca)$.

In the semigroup, the palindromic length can clearly be computed, and fast algorithms exist for that. In the group, it is decidable if the palindromic length is equal to 2 and perhaps to 3, but the amount of cases to be considered for the length equal to 3 suggests that even the case of 4 is too complicated to be treated by the existing method. So, the general question of decidability remains open, and I am going to discuss it.

Tarek Sellami

Isolated points in Heinis spectrum

Complexity function and Heinis Spectrum:

Definition

The complexity function of an infinite word u is the map from $\mathbb{N}$ to $\mathbb{N}^*$ defined by:

$$p_u(n) = |\mathcal{L}_n(u)|.$$

Definition

The Heinis spectrum denoted by H and defined as follows:

$$H = \{(\alpha, \beta) : u \in A^{\mathbb{N}} \text{ s.t. } u \text{ is recurrent}\} \subseteq (\mathbb{R} \cup \{+\infty\})^2.$$

where $\alpha = \liminf_{n\to\infty} \frac{p_u(n)}{n}$ *and* $\beta = \limsup_{n\to\infty} \frac{p_u(n)}{n}$ *for every* $u \in A^{\mathbb{N}}$.

Examples:

- $(\alpha, \beta) = (1, 1)$ for sturmian words.
- $(\alpha, \beta) = (2, 2)$ for Arnoux-Rauzy words.
- $(\alpha, \beta) = (3, \frac{10}{3})$ for Thue-morse word.

Theorem

The point $(\alpha, \beta) = (\frac{3}{2}, \frac{5}{3})$ *is an isolated point in* H *and it can be obtained from the fixed point* u *of the following substitution defined over the alphabet* $\{a, b\}$ *by:*

$$\begin{aligned} \tau : a &\longrightarrow bb \\ b &\longrightarrow ba. \end{aligned}$$

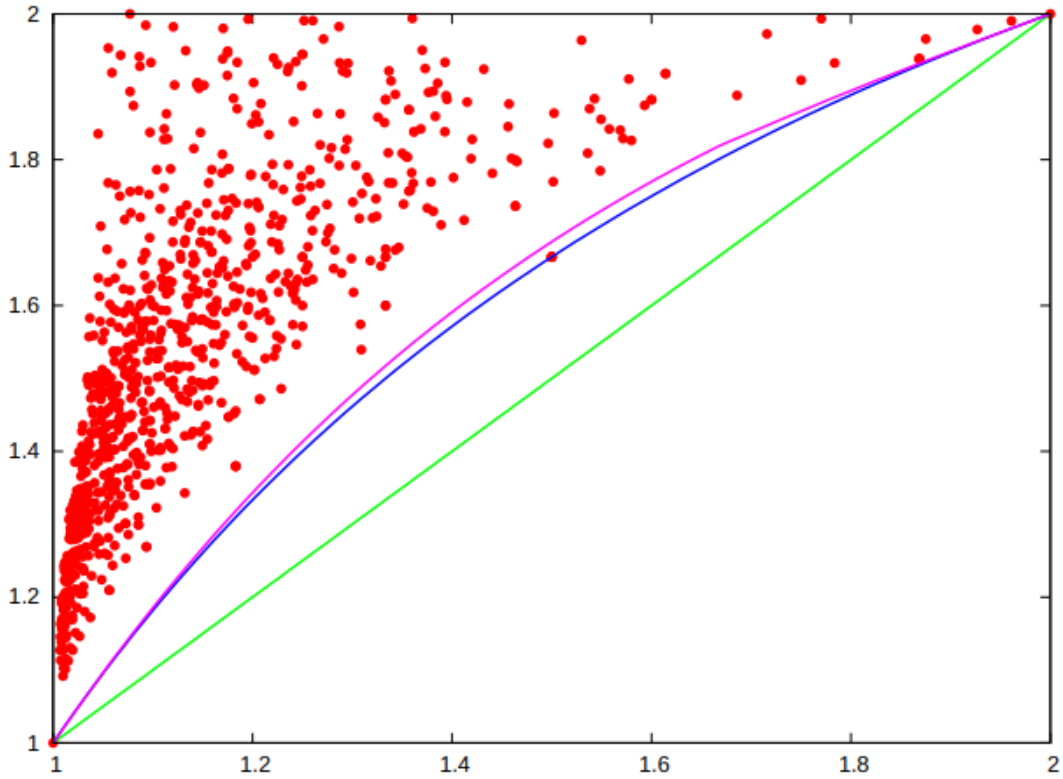


Figure: Some points of Heinis spectrum. [J. Cassaigne]

Theorem

Let u be an infinite word with infinitely many Rauzy graphs of type S, and let $n_0, k \in \mathbb{N}$. Assume that, for every $n \geq n_0$, if Γ_n is a graph of type S, then it undergoes the evolution $O_{2,x} \circ O_{1,y}^k$ (resp. $O_{1,y}^k \circ O_{2,x}$). Then:

$$(\alpha, \beta) = (1 + \frac{2}{(k+1)(\sqrt{\Delta}+k+1)}; 1 + \frac{5k+\sqrt{\Delta}+9}{6k^2+22k+18}),$$

where $\Delta = k^2 + 2k + 9$.

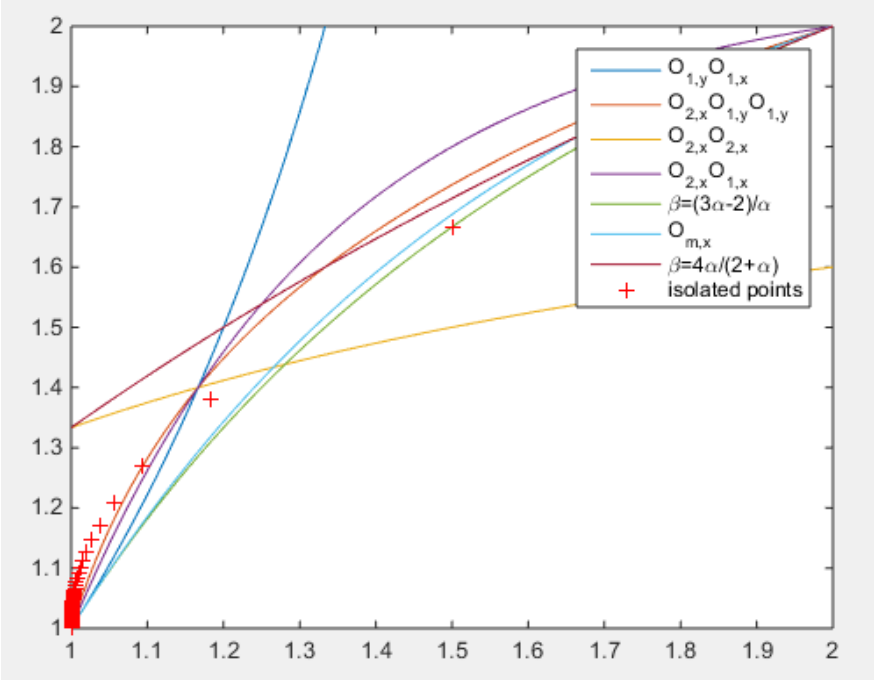


Figure: Potential Isolated Points within H.

Question:

- Are the points (α, β) obtained with $O_{2,x}O_{1,y}^{k}$ isolated points in H?
- A similar calculation can be done for $O_{m,x}O_{1,y}^{k}$ with $m \geq 3$ to obtained (α, β). Are these points also isolated in H?

Thank you for your attention!

Definition

*For all $n \in \mathbb{N}$, the **Rauzy graph** of order n of u, denoted by $\Gamma_n(u)$ (or Simply Γ_n), is the labelled directed graph, such that:*

- *Its vertices are the factors of length n of u.*
- *There exists an edge from a vertex w to a vertex v if and only if there exist $a, b \in A$ such that $wa = bv \in \mathcal{L}_{n+1}(u)$. The letter a is called label of the edge from w to v, and we note $w \xrightarrow{a} v$.*

Notations

- *$S(n,x,y)$ is the graph of order n with a bispecial factor and x, y are the words which are the labels of the two loops.*
- *$T(n,x,y,z)$ is the graph of order n with only one right and one left special factor, x, y and z are the words labelling the branches.*

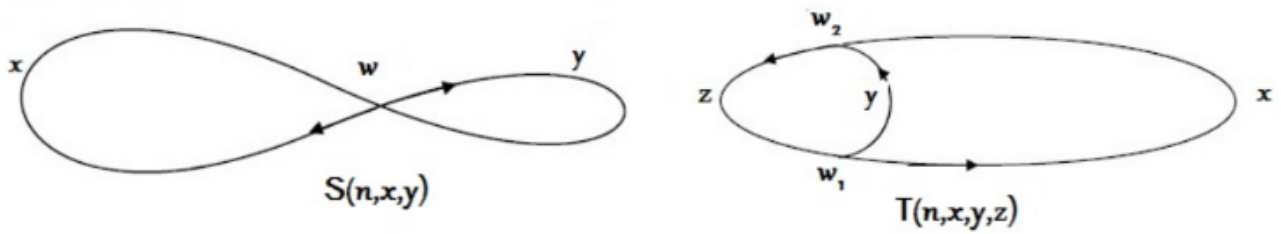


Figure: Sturmian graphs.

Definition

We note $O_{1,x}$ and $O_{1,y}$ two types of evolutions between two graphs of type S defined by:

- $O_{1,x}(S(n,x,y)) = (G_1, G_2, ..., G_{|x|})$, *where:*

$$\begin{aligned} G_i &= T(n+i, yx_{[\![0,i[\![}, x_{[\![0,i[\![}, x_{[\![|x|-i,|x|[\![}) \text{ if } 1 \leq i < |x|, \text{ and,} \\ G_{|x|} &= S(n+|x|, yx, x), \text{ otherwise.} \end{aligned}$$

- $O_{1,y}(S(n,x,y)) = (G_1, G_2, ..., G_{|y|})$, *where:*

$$\begin{aligned} G_i &= T(n+i, xy_{[\![0,i[\![}, y_{[\![0,i[\![}, y_{[\![|y|-i,|y|[\![}) \text{ if } 1 \leq i < |y|, \text{ and,} \\ G_{|y|} &= S(n+|y|, xy, y), \text{ otherwise.} \end{aligned}$$

For $m \geq 2$, we define the evolution $O_{m,x}$ on graphs of type S by: $O_{m,x}(S(n,x,y)) = (G_1, ..., G_{|x|})$ where $|x| > (m-1)|y|$ (otherwise not defined), and for wich the next Rauzy graph of type S is:

$$G_{|x|} = S(n+|x|, y^m x, x).$$

Lemmas:

Let u be an infinite word with infinitely many Rauzy graphs of type S, and let $n_0, k \in \mathbb{N}$.
If for every $n \geq n_0$, if Γ_n is a graph of type S, then it undergoes the evolution:

- $O_{m,x}$ with $m \geq 3 \Longrightarrow \beta \geq \dfrac{5\alpha^2 - 3\alpha}{2\alpha^2 - \alpha + 1}$.
- $O_{2,x}O_{1,x} \Longrightarrow \beta \geq \dfrac{4\alpha^2 - 3\alpha}{2\alpha^2 - 2\alpha + 1}$.
- $O_{2,x}O_{1,y} \Longrightarrow \beta \geq \dfrac{(6\alpha - 5)\alpha}{2\alpha^2 - 1}$.
- $O_{1,x}O_{1,y} \Longrightarrow \beta \geq \dfrac{\alpha}{2 - \alpha}$.

Using the same idea, we are trying to prove that (α, β) obtained before (the case of $O_{2,x}O_{1,y}^k$ are isolated in H, and we start by the case of $k = 1$:

Lemmas:

Let u be an infinite word with infinitely many Rauzy graphs of type S, and let $n_0 \in \mathbb{N}$.
If for every $n \geq n_0$, if Γ_n is a graph of type S, then it undergoes the evolution:

- $O_{m,x}$ with $m \geq 3 \Longrightarrow \beta \geq \dfrac{5\alpha^2 - 3\alpha}{2\alpha^2 - \alpha + 1}$.
- $O_{2,x}O_{1,x} \Longrightarrow \beta \geq \dfrac{4\alpha^2 - 3\alpha}{2\alpha^2 - 2\alpha + 1}$.
- $O_{1,x}O_{1,y} \Longrightarrow \beta \geq \dfrac{\alpha}{2 - \alpha}$.
- $O_{2,x}O_{2,x} \Longrightarrow \beta \geq \dfrac{4\alpha}{1 + 2\alpha}$.
- $O_{2,x}O_{1,y}O_{1,y}$: $\beta \geq \dfrac{9\alpha^2 - 8\alpha}{3\alpha^2 - 2}$.

Savinien Kreczman

Context-freeness of languages derived from a greedy numeration system

Context-freeness of languages derived from a greedy numeration system

Start with a linear recurrence sequence $(U_n)_{n\in\mathbb{N}}$.
A numeration system can be defined from such a sequence using a greedy algorithm. The language of this numeration system is

$$\begin{aligned} L_U &= \{\mathrm{rep}(n) : n \in \mathbb{N}\} \\ &= \{w_\ell \cdots w_0 : \forall j,\ w_{j-1} \cdots w_0 \preccurlyeq \mathrm{rep}(U_j - 1)\} \end{aligned}$$

We let $\mathrm{Maxlg}\, L_U = \{\mathrm{rep}(U_j - 1) : j \in \mathbb{N}\}$.
The sequence U_n has a *dominant root* if $\frac{U_n}{U_{n-1}}$ has a limit.

Context-freeness of languages derived from a greedy numeration system

Question

Does there exist a sequence U with a dominant root, such that $\mathrm{Maxlg}\, L_U$ is context-free but not regular?

Question

Is it true that $\mathrm{Maxlg}\, L_U$ context-free implies L_U context-free? (True for regularity)

Question

Is context-freeness even "the right" extension of regular languages to look at?

A nondominant example

Consider the sequence U given by

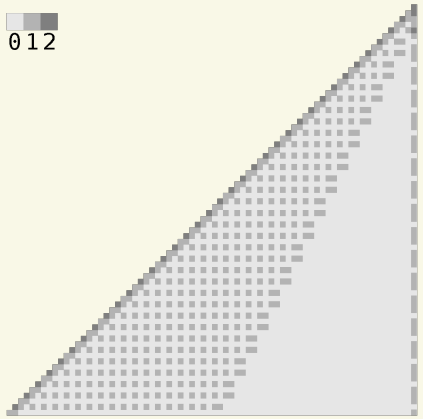
$$U_{n+6} = 6U_{n+4} - 9U_{n+2} + 4U_n$$

and the initial conditions

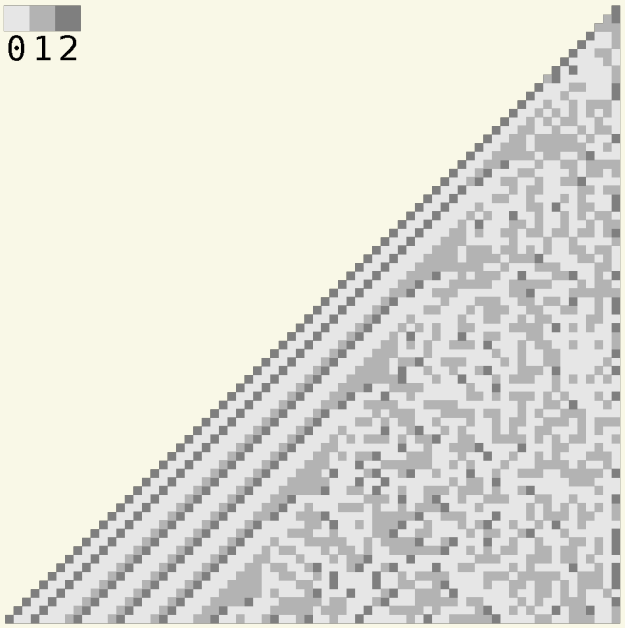
$$1, 3, 6, 17, 25, 73.$$

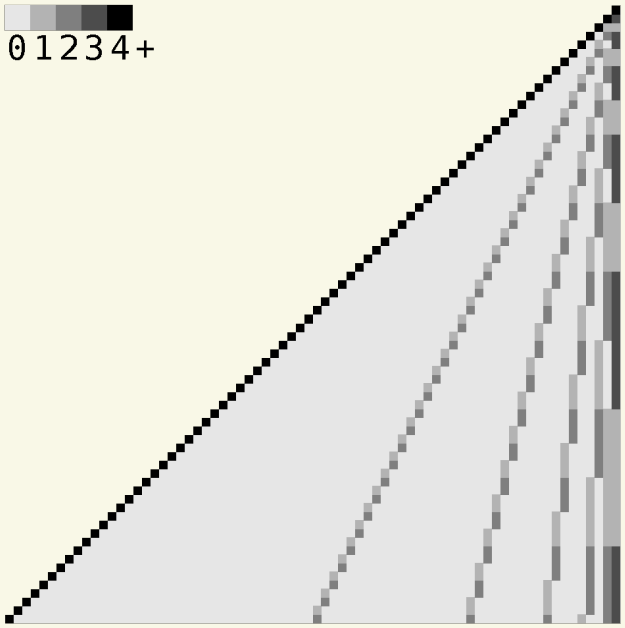
The sequence continues $100, 297, 399, \ldots$ and we have for instance

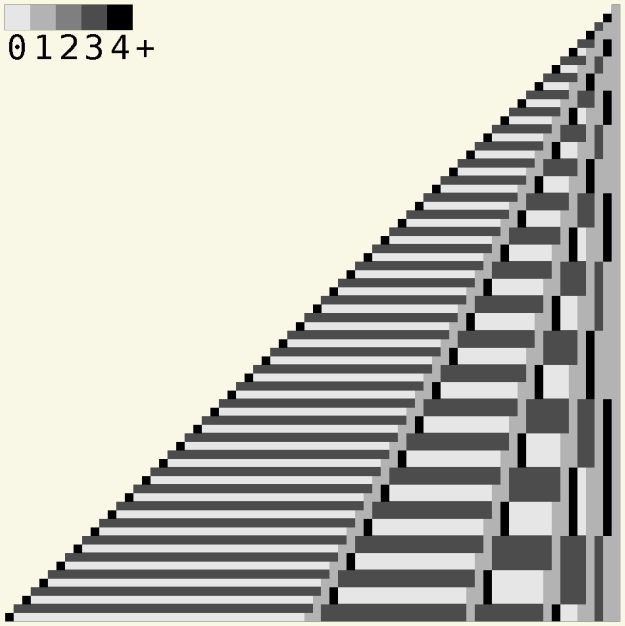
$$296 = 2 \cdot 100 + 73 + 17 + 6, \; \mathrm{rep}(296) = 2101100$$

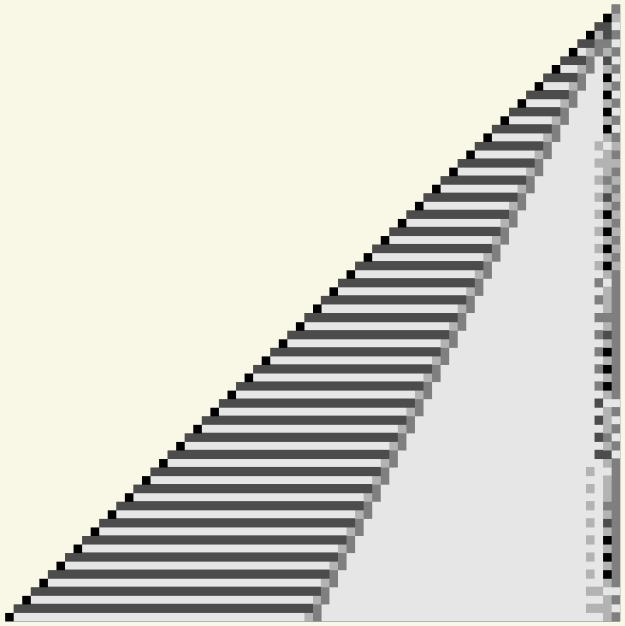


Some pictures









Bartłomiej Pawlik

Shuffle squares

Reconstructing Shuffle Squares from Cuts

Jarosław Grytczuk, Bartłomiej Pawlik, Mariusz Pleszczyński

Tangram is a word in which every letter occurs an even number of times.

Some binary shuffle squares:

$$\begin{array}{ccc} 001001 & 011000 & 001100 \\ 001010 & 011011 & 010111 \end{array}$$

Previous conjecture (2023):

For every binary tangram W, there exists a factorisation $W = AB$ such that BA is a shuffle square.

$$100010 = 10|0010 \;\rightarrow\; 0010|10 = 001010$$

Counterexample (2024): $0^5 1 0^2 1^4 0^4 1^3 0 1^4$

Modified conjecture (2024):

For every $k \geqslant 2$ and every tangram W over A_k there exists factoristation of W into $(k+1)$ factors such that the concatenation of some permutation of these factors forms a shuffle square.

$$000001001111000011101111 = 0000|01|001111000011101111 \;\rightarrow$$

$$\rightarrow\; 0000|001111000011101111|01 = 000000111100001110111101$$

Collecting the shuffle squares

Bartłomiej Pawlik

For any natural language, construct the longest sentence in that language, which is a sh square but not a square. (thus the words like HOTSHOTS are not allowed)

Known words: *håhåjaja* (Swedish), *tuteurer* (French)

HÅHÅJAJA, TUTEURER

Known sentences (Polish):

- Andrzej Ruciński, July 2023, length 12:

  Nina, mima mama. (*Nina, mum of the mime.*)

  NINAMIMAMAMA

- Jan Szejko, August 24$^{\text{th}}$ 2024, length 22:

  Dziedziczenie czekanika. (*Inheritance of the small ice axe.*)

  DZIEDZICZENIECZEKANIKA

Gabriele Fici

Cyclic equalizability of words

Given two words $u_1...u_n$ and $v_1...v_n$, and $i \leq n$, an insertion of letter a at position i produces $u_1...u_{i-1}au_i...u_n$ and $v_1...v_{i-1}av_i...v_n$.

Given two binary words u and v with the same Parikh vector, it is always possible to find a sequence of insertions that produces two conjugated words (Shinagawa & Nuida, FCT 2025).

Is it possible to do the same for three binary words?