Purely automatic sequences with the uniform distribution property

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Presentation plan

Uniformly distributed sequences

- Let $W = W[0]W[1]\ldots W[n]\ldots$ be a sequence over the alphabet $\{1, \ldots, M\}$.
- The sequence W is called uniformly distributed if for all pairs of integers $a, b \in \mathbb{N}$ such that $a \geq 1$, $0 \leq b < a$ and all $m \in \{1, \ldots, M\}$, one has

$$d_{W,a,b}(m) = \lim_{n\to\infty} \frac{\#\{k | W[ak+b] = m, 0 \leq ak+b < n\}}{n} = \frac{1}{aM}.$$

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First result

Theorem (Gelfond,68)

For integers $q, M, d, b, a \in \mathbb{N}$ *such that* $q, M, a \geq 2$ *and* $\gcd(M, q-1) = 1$,

$$\lim_{n \to \infty} \frac{\#\{k | s_q(ak+b) \equiv d \bmod M, 0 \leq ak+b < n\}}{n} = \frac{1}{aM},$$

where $s_q(n)$ *is the sum of all digits in the* q*-expansion of* n *for all* $n \in \mathbb{N}$.

(Purely) automatic sequences

- let A and B be two finite alphabets.
- A sequence V is called automatic over A if :
  - there exists a l-uniform morphism $\phi : B^* \to B^*$, i.e. for all $x \in B$,
  $$|\phi(x)| = l;$$
  - there exists a letter a such that $\phi(a) = aw$;
  - there exists a 1-uniform morphisme $\psi : B \to A$, which is also called a coding function ;
  - $$V = \lim_{n\to\infty} \psi(\phi^n(a)).$$
- A sequence V is called purely automatic over A if $B = A$ and $\psi = Id$.

Some examples

- Let $\phi : \{1,2,3\}^* \to \{1,2,3\}^*$ such that $\begin{cases} \phi(1) = 132; \\ \phi(2) = 312; \\ \phi(3) = 213. \end{cases}$
- Then

$$\begin{aligned} U &= \lim_{n\to\infty} \phi^n(1) \\ &= 132\ldots; \\ &= 132213312\ldots; \\ &= 132213312312132213213132312\ldots; \\ &= \ldots \end{aligned}$$

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- Let $\psi : \{1,2,3\}^* \to \{A,B\}^*$ such that
$$\begin{cases} \psi(1) = \psi(3) = A; \\ \psi(2) = B. \end{cases}$$
- Then
$$\begin{aligned} V &= \psi(U) \\ &= AABBAAAABAABAABBAABAAAABAAB\ldots; \end{aligned}$$
is automatic (and maybe also purely automatic!)

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Questions

- The sum-of-digits functions studied by Gelfond are all automatic, but not all of them are purely automatic.
- Question 1 : When will an automatic sequence be uniformly distributed ?
- Question 2 : When will a purely automatic sequence be uniformly distributed ?

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- Question 1 : When will an automatic sequence be uniformly distributed ?
- Question 2 : When will a purely automatic sequence be uniformly distributed ?

Set up

- Let $M, L \in \mathbb{N}$ such that $M > 1, L > 1$.
- Let ϕ be a L-uniform morphism over $\{1, \ldots, M\}$, i.e. $|\phi(x)| = L$ for all $x \in \{1, \ldots, M\}$.
- Suppose $\phi(1) = 1w$ for some lenth-$L-1$ word w.
- Let $W = \lim_{n\to\infty} \phi^n(1)$.

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First characterization

Proposition

If W satisfies the uniform distribution property, then, necessarily, for each $0 \leq l < L$, $\phi(1)[l], \phi(2)[l], \cdots, \phi(M)[l]$ are pairwise distinct.

$$\begin{cases} \phi(1) = 132; \\ \phi(2) = 321; \\ \phi(3) = 213. \end{cases} \qquad \begin{cases} \phi'(1) = 132; \\ \phi'(2) = 123; \\ \phi'(3) = 213. \end{cases}$$

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- Let $W = \lim_{n \to \infty} \phi^n(1)$.
- For $0 \leq l < L$, let $P_l : \{1, \ldots, M\} \to \{1, \ldots, M\}$ be a permutation such that $P_l(m) = \phi(m)[l]$ for all $m \in \{1, \ldots, M\}$.
- Let M_{P_l} be the matrix representation of P_l.

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- Let M_{P_l} be the matrix representation of P_l.

Approach by substitution

- Let $M = 2$, $\phi : \{1,2\}^* \to \{1,2\}^*$ satisfying

$$\begin{cases} \phi(1) = 1121; \\ \phi(2) = 2212. \end{cases}$$

Thus, $L = 4$.

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- Let us compute the densities of 1 and 2 along the arithmetic progression $3x + i$, for $i = 0, 1$ or 2.

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\begin{aligned}
&\#\{k|U[3k]=1,0\leq 3k<4^6\}\\
=&\#\{k|U[12k]=1;0\leq 12k<4^5\}\\
+&\#\{k|U[12k+3]=1,\leq 12k+3<4^5\}\\
+&\#\{k|U[12k+6]=1;\leq 12k+6<4^5\}\\
+&\#\{k|U[12k+9]=1,0\leq 12k+9<4^5\}\\
=&\#\{k|U[3k]=P_0^{-1}(1),0\leq 3k<4^5\}\\
+&\#\{k|U[3k]=P_3^{-1}(1),0\leq 3k<4^5\}\\
+&\#\{k|U[3k+1]=P_2^{-1}(1),0\leq 3k+1<4^5\}\\
+&\#\{k|U[3k+2]=P_1^{-1}(1),0\leq 3k+2<4^5\}.
\end{aligned}
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Approach by substitution

- Let $v_{a,b,m,i,j}$ denote the number of m lying on the arithmetic progression $ax+b$ with an index bounded between iL^j and $(i+1)L^j$.
- $v_{3,0,1,0,6} = v_{3,0,1,0,5} + v_{3,0,1,0,5} + v_{3,1,2,0,5} + v_{3,2,1,0,5}.$
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- Set

$$M_{\phi,3} = \begin{bmatrix} 2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \\ 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 & 0 & 2 \end{bmatrix}$$

- For all $i, j \in \mathbb{N}$, one has

$$\begin{bmatrix} v_{3,0,1,i,j+1} \\ v_{3,0,2,i,j+1} \\ v_{3,1,1,i,j+1} \\ v_{3,1,2,i,j+1} \\ v_{3,2,1,i,j+1} \\ v_{3,2,2,i,j+1} \end{bmatrix} = M_{\phi,3} \begin{bmatrix} v_{3,0,1,i,j} \\ v_{3,0,2,i,j} \\ v_{3,1,1,i,j} \\ v_{3,1,2,i,j} \\ v_{3,2,1,i,j} \\ v_{3,2,2,i,j} \end{bmatrix}.$$

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Proposition

The sequence U satisfies

$$\frac{\#\{k|U[3k+i]=j;0\le 3k+i<n\}}{n}=\frac{1}{6}$$

if and only if the matrix $M_{\phi,3}$ is irreducible, i.e., for any $1\le r,s\le kM$, there exists an integer t such that $M_{\phi,3}^t(r,s)>0$.

Theorem

In general, the sequence W is uniformly distributed if and only if the matrix $M_{\phi,k}$ is irreducible for all $k\in\mathbb{N}$.

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Theorem

In general, the sequence W is uniformly distributed if and only if the matrix $M_{\phi,k}$ is irreducible for all $k \in \mathbb{N}$.

Construction of $M_{\phi,k}$

- Recall that W is a fixed point of a L-uniform morphism over the alphabet $\{1, \ldots, M\}$.
- For $0 \leq l < L$, P_l is the permutation $P_l(i) = \phi(i)[l]$ and M_{P_l} is the matrix presentation of P_l.
- Then the matrix $M_{\phi,k}$ is a matrix of size $Mk \times Mk$.

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Construction of $M_{\phi,k}$

- Filling the matrix $M_{\phi,k}$ by blocks of size $M \times M$.
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$$M_{\phi,k} = \begin{bmatrix} M_{P_0} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

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- If $k < L$.
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$$M_{\phi,k} = \begin{bmatrix} M_{P_0} + M_{P_{k+1}} & \dots & \\ M_{P_1} + M_{P_{k+2}} & \dots & \\ M_{P_2} + M_{P_{k+3}} & \dots & \\ \dots & \dots & \\ \dots & \dots & \\ M_{P_k} + \dots & \dots & \end{bmatrix}$$

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$L = 4$.

- Let $U = \lim_{n\to\infty} \psi^n(1)$.
- Thus,

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Approach by substitution

To compute $M_{\phi,3}$

$$M_{\phi,3} = \begin{bmatrix} M_{P_0} + M_{P_3} & M_{P_2} & M_{P_1} \\ M_{P_1} & M_{P_0} + M_{P_3} & M_{P_2} \\ M_{P_2} & M_{P_1} & M_{P_0} + M_{P_3} \end{bmatrix}$$

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Thus,

$$M_{\phi,3} = \begin{bmatrix} 2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \\ 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 & 0 & 2 \end{bmatrix}$$

Approach by substitution

Thus,

$$M_{\phi,3} = \begin{bmatrix} 2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \\ 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 & 0 & 2 \end{bmatrix}$$

Reduced matrices

The reduced matrix $m_{\phi,k}$ is a matrix of the size $k \times k$ satisfying

$$m_{\phi,k}(i,j) = \begin{cases} 1 & \text{if } (M_{\phi,k}(p,q))_{\substack{iM+1 \le p < i(M+1) \\ jM+1 \le q < j(M+1)}} \neq \mathbf{0}; \\ 0 & \text{otherwise.} \end{cases}$$

Example

$$M_{\phi,3} = \begin{bmatrix} M_{P_0} + M_{P_3} & M_{P_2} & M_{P_1} \\ M_{P_1} & M_{P_0} + M_{P_3} & M_{P_2} \\ M_{P_2} & M_{P_1} & M_{P_0} + M_{P_3} \end{bmatrix}$$

$$m_{\phi,3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Example

$$M_{\phi,5} = \begin{bmatrix} M_{P_0} & M_{P_1} & M_{P_2} & M_{P_3} & 0 \\ M_{P_1} & M_{P_2} & M_{P_3} & 0 & M_{P_0} \\ M_{P_2} & M_{P_3} & 0 & M_{P_0} & M_{P_1} \\ M_{P_3} & 0 & M_{P_0} & M_{P_1} & M_{P_2} \\ 0 & M_{P_0} & M_{P_1} & M_{P_2} & M_{P_3} \end{bmatrix}$$

$$m_{\phi,5} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

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$$M_{\phi,5} = \begin{bmatrix} M_{P_0} & M_{P_1} & M_{P_2} & M_{P_3} & 0 \\ M_{P_1} & M_{P_2} & M_{P_3} & 0 & M_{P_0} \\ M_{P_2} & M_{P_3} & 0 & M_{P_0} & M_{P_1} \\ M_{P_3} & 0 & M_{P_0} & M_{P_1} & M_{P_2} \\ 0 & M_{P_0} & M_{P_1} & M_{P_2} & M_{P_3} \end{bmatrix}$$

$$m_{\phi,5} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Some properties

Proposition

The matrices $m_{\phi,k}$ only depend on the length of ϕ.

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The matrices $m_{\phi,k}$ are always irreducible (and primitive).

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A necessary condition

Theorem

If ϕ is a L-uniform morphism satisfying that

- *The maps $P_l : i \to \phi(i)[l]$ are all permutations over $\{1, \ldots, M\}$ for $0 \le l < L$;*
- *p_{L-1} is a non-trivial rotation,*

then $M_{\phi,k}$ is irreducible for all k. Consequently, any fixed point W of ϕ is uniformly distributed.

Proof

Fact :

$$M_{\phi,k} = \begin{bmatrix} & & & & & \cdots \\ & & & & & \cdots \\ & & & & & \cdots \\ & & & & & \cdots \\ & & & & & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & M_{P_L}+? \end{bmatrix}$$

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Proof

Let $1 \leq i, j \leq kM$.

Since $m_{\phi,k}$ is irreducible, there exist t_1, t_2 such that

$$m_{\phi,k}^{t_1}(i,k) > 0, m_{\phi,k}^{t_2}(k,j) > 0$$

.

There exist $(k-1)M + 1 \leq i', j' \leq kM$ such that

$$M_{\phi,k}^{t_1}(i,i') > 0, M_{\phi,k}^{t_2}(j',j) > 0$$

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There exists t_3 such that

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Thus,

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Questions

- Is there a characterization of the uniform distribution property in terms of the combinatorial properties of the generating morphism ?
- Is it true that a purely automatic sequence has the uniform distribution property if and only if
  - it is aperiodic,
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Thank you for your attention.